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RESEARCHES INTO THE THEORY OF PROBABILITY

BY

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I. Introduction.

Among newer investigations into the theory of probability I know none more important than those of PEARSON in his admirable series of »Contributions to the mathematical theory of evolution». The numerous school of biologists that has grown up during the last ten years, which has applied his methods to fundamental problems in botany and zoology, has richly demonstrated the importance of these methods for biology and shown the possibility of basing the science of life on exact mathematical methods. The branch of mathematics that is here in the first place needed is the theory of probability. For this reason PEARSON was obliged, in attacking the problem of evolution from a mathematical point of view, to solve some important problems in this theory, that had not to that time been sufficiently dealt with. He has solved a great part of these problems. Others remained unsolved or only partially solved. The object of the present investigation is to treat some of these problems, which are of great importance not only to biology, but to all sciences based on observations of nature. I should be glad if the results obtained will contribute to further develop the line of research laid out by PEARSON and his school.

Taking an arbitrary individual in the living nature — a man, an animal, a plant — it will generally be found impossible to find out another individual in all respects identical to the one first chosen. If the difference is great, we say that the two objects belong to different orders, classes, species, subspecies a. s. o., but it is impossible to carry the classification so far, that the differences between the individuals of the same sub-class would disappear. Nevertheless there is something that rightly may be named classe, species a. s. o. of individuals, though the strict definition of these terms is difficult and scarcely can be made without employing mathematical methods.

Let us consider a number of individuals all belonging to the same species, by which term we mean for the moment the narrowest group in the classification of the objects in question. We take into consideration a certain character of these individuals, and assume that this character may be measured as to its magnitude or intensity, so that the measurements are expressed through numbers. Generally the character may vary continuously, and its true value in each individual can then only be measured approximately as the height of a man. In some cases the magnitude determinations of a character are expressed exactly in numbers, as the numbers of petals in a flower. In either case we generally find that the character varies from one individual to another. In known manner the characters continuously varying may be treated in the same manner as those expressible in integers and we assume that, expressed in a certain unit, the character x may assume all, or at least some, of the integer values

$$0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots$$

Counting the number — y — of individuals having a certain magnitude in respect to the character in consideration, we obtain what is called a frequency-table or — graphically — a *frequency-curve.*

What is the form of this curve?

The question seems at the first glance to be somewhat vague, if not unanswerable. Nevertheless experience has shown, that this curve really has a certain form, which may be mathematically defined, and, what is still more astonishing; that the parameters necessary to mathematically define a certain frequency-curve are generally very few in number. Very often 3 parameters suffice for representing, with satisfactory approximation, a collection of thousands of individuals. It is the duty of the mathematician to find the equation of this curve. As to the search for the hypotheses necessary to declare the origin of the frequency-curve, the mathematician and the observer of the nature must work together.

These hypotheses may be formulated in different ways. The question is to find a hypothesis that will suffice for declaring all the different forms in which the frequency-curves can occur. In searching for such a hypothesis we are aided by the methods used in solving an astronomical problem of similar character. I mean the explanation of the errors of observation.

According to Hagen and Bessel, who have given the best explanation of this difficult problem, an error of observation may be considered as the *sum of a great many very small elementary errors.* Let us suppose the question is to determine the siderial time through meridian observations of stars. If the transit instrument were installed exactly in the meridian, if the right ascensions of the stars were exactly known, if the meteorological conditions of the atmosphere were known in all details, if the physiological state of the observer at all observations were unaltered and if all other circumstances that may have influence on the result were the same at all observations, it is clear that we should obtain full agreement between the observed values of the clock-correction. The true conditions, however, are somewhat different from this ideal state. The adjustment of the instrument is not fully correct, the coordinates of the stars are affected by small errors, the temperature, pressure and other conditions of the atmosphere differ from one moment to another a. s. o. Each error of observation therefore may be considered as the sum of a multitude of small errors, derived from equally many independent *sources*. The law according to which the errors of each source varies may be different for each source and must *a priori* be considered as unknown.

In essentially the same manner we can declare the variation of the characters in biology. Consider, for instance, the stature of a group of adult men. If all men in the group be supposed to possess identically similar ancestors, if they have enjoyed identically the same education, the same food, the same climatical influences, if all other circumstances that may have some influence on the stature of the man were identically similar for all men in the group, we must conclude that the length of the stature of all these men must be the same, as truly as the effect is determined from the cause. The differences in ancestral heredity, in education, in food a. s. o. for a group of men may be considered as different sources of error as to the stature of these men. Each source of error may cause a positive or negative »elementary error» in the length; and through the addition of these small quantities the resulting deviation in the length of an individual from the supposed ideal length is obtained. Obviously the number of the sources of these elementary errors must be considered as very great, if not infinite.

This manner considering things seems to be very plausible. Meanwhile a new difficulty appears, a difficulty of a mathematical character, which seems to make the problem almost unsoluble. The number of the sources of error that each give elementary errors is supposed to be very great and each source has its own law of error, which must be considered as unknown. How great is the sum of all these elementary errors? The problem is very difficult, but it has been attacked and in principle solved by LAPLACE in his great work »Théorie analytique des probabilités» (1820). In two memoirs on the law of errors (Meddelanden från Lunds observatorium N:ris 25 och 26) I have discussed the problem, and shown some consequences that may be drawn from the results of LAPLACE.

These consequences are the following ones.

A frequency curve may possess one of the following two forms:

Type A. If the frequency curve is defined by the equation $y = F(x)$, where x is the measure of the character in question, and y its frequency, and we put

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-b)^2}{2\sigma^2}},$$

designating by b and σ two parameters, which must be duly determined, we can represent the frequency curve of type A through the equation

$$F(x) = A_0\,\varphi(x) + A_3\,\varphi^{\mathrm{III}}(x) + A_4\,\varphi^{\mathrm{IV}}(x) + \ldots,$$

where A_0, A_3, A_4, ... are coefficients independent of x.

Type B. The frequency curve of the second form may be expressed with the help of the auxiliary function

$$\psi(x) = \frac{e^{-\lambda} \sin \pi x}{\pi} \left[\frac{1}{x} - \frac{\lambda}{\underline{|1}\,(x-1)} + \frac{\lambda^2}{\underline{|2}\,(x-2)} - \frac{\lambda^3}{\underline{|3}\,(x-3)} + \ldots \right],$$

where λ is a parameter, and the general form of $F(x)$ is then

$$F(x) = B_0\, \psi(x) + B_1\, \Delta\psi(x) + B_2\, \Delta^2\, \psi(x) + \ldots,$$

where B_0, B_1, B_2, .. are coefficients independent of x.

Beyond these two forms no other frequency curves can occur, except those obtained through a superposition (addition) of several curves of the types A and B.

I will in this memoir more fully discuss these two forms of the frequency curve.

As to the conditions for the rise of these two types, it may for the present suffice to observe that type B arises, if the probability of a deviation from the »ideal» value of a character, caused by each single source of error is very small, whereas those sources of error, that possess an equal or nearly equal probability for such values of the character as lie in the neighbourhood of the »ideal» one give rise to a frequency curve of the first type.

By ideal value of the character here is meant such a value as would arise if all sources of error that may influence on the character had their most probable state. For the more precise formulation of the conditions for the two forms I refer to the mathematical investigation in the memoirs cited. It must be remarked that it is possible to pass continuously from one form to the other.

II. Type A of frequency curves.

Let x be the value of a character and $F(x)\,dx$ the frequency of those values that lie between x and $x + dx$. The frequency $F(x)$ is represented by means of the equation

$$F(x) = A_0\, \varphi(x) + A_3\, \varphi^{\mathrm{III}}(x) + A_4\, \varphi^{\mathrm{IV}}(x) + \ldots, \tag{1}$$

where

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-b)^2}{2\sigma^2}}. \tag{1*}$$

The quantities b, σ, A_0, A_3; A_4, ... are dependent on the form of the equation $y = F(x)$. The formulæ for determining these quantities have been given in my treatise »Über die Darstellung willkürlicher Functionen» (»Meddelanden» N:o 27).

Choosing the origin of the x-coordinates *arbitrarily*, we put

$$\mu'_s = \int_{-\infty}^{+\infty} x^s F(x)\, dx. \tag{2}$$

On the other side we put

$$\mu_s = \int_{-\infty}^{+\infty} (x-b)^s F(x)\, dx, \tag{2*}$$

so that

$$\mu_s = \mu'_s - \binom{s}{1} b\, \mu'_{s-1} + \binom{s}{2} b^2 \mu'_{s-2} - \binom{s}{3} b^3 \mu'_{s-3} + \dots$$

where $\binom{s}{1}$, $\binom{s}{2}$, $\binom{s}{3}$ designate the binomial coefficients.

If the quantity b is known, we know also the values of μ_0, μ_1, μ_2, μ_3, . . Now b is given by the equation

$$\mu'_0\, b = \mu'_1. \tag{3}$$

We then have

$$\mu_0\, \sigma^2 = \mu_2, \tag{3*}$$

and the quantities A_0, A_3, A_4, . . . have the values

$$\begin{aligned} A_0 &= \mu_0, \\ \lfloor 3\, A_3 &= -\mu_3, \\ \lfloor 4\, A_4 &= \mu_4 - 3\sigma^4 \mu_0, \\ \lfloor 5\, A_5 &= -\mu_5 + 10\sigma^2 \mu_3, \\ \lfloor 6\, A_6 &= \mu_6 - 15\sigma^2 \mu_4 + 15\sigma^6 \mu_0, \end{aligned} \tag{4}$$

The quantities μ_1, μ_2, μ_3, . . . are named the moments, taken in respect to (or about) the point b, of the curve $y = F(x)$ of the first, second, third, . . . order. When these quantities are calculated, it is easy to calculate the values of the coefficients A_0, A_3, A_4, . . . according to the formulæ (4).

As to σ it is named by English writers on probability the *standard deviation.* German mathematicians generally call it *mean deviation* or *mean error.* As to $\varphi(x)$, it is the form of the *probability function* generally used by Pearson. I find that this form is to be preferred before the usual Gaussian form

$$\varphi(x) = \frac{k}{\sqrt{\pi}}\, e^{-k^2 (x-b)^2},$$

where k is called the *measure of precision.* The difference is naturally only a formal one, but σ, being a length (supposing x to be considered as a length), is easier to conceive than the quantity k. I will in this connection remark that the so-called *probable* error may without regret be removed from the practical applications of the theory of probability, as the standard (mean) deviation says all that is wanted from the calculus in the respect that here is concerned.

The values of the probability function $\varphi(x)$ are most conveniently tabulated by SHEPPARD (»Biometrica» 1903). The argument of these tables are the quotient $(x-b):\sigma$. In the same memoir also the values of the probability integral, that is of the integral

$$\int \varphi(x)\,dx$$

are given in a similar manner.

As to the form of the derivated functions of φ I remind of the relation

$$\varphi^{s}(x) = R_s(x)\,\varphi(x),$$

where $R_s(x)$ is a whole rational function (i. e. a polynom) of x of the degree s. For the lowest values of s we have

$$\begin{aligned}
R_0 &= 1,\\
\sigma^2\, R_1 &= -(x-b),\\
\sigma^4\, R_2 &= +(x-b)^2 - \sigma^2,\\
\sigma^6\, R_3 &= -(x-b)^3 + 3\,\sigma^2\,(x-b),\\
\sigma^8\, R_4 &= +(x-b)^4 - 6\,\sigma^2\,(x-b)^2 + 3\,\sigma^4,\\
\sigma^{10}\, R_5 &= -(x-b)^5 + 10\,\sigma^2\,(x-b)^3 - 15\,\sigma^4\,(x-b),\\
\sigma^{12}\, R_6 &= +(x-b)^6 - 15\,\sigma^2\,(x-b)^4 + 30\,\sigma^4\,(x-b)^2 - 15\,\sigma^6,\\
&\;\;\cdots
\end{aligned}$$

Hence we find that $\sigma^s R_s$ is a function only dependent on the quotient $(x-b):\sigma$. As the product $\sigma\varphi(x)$ also depends only on the same quotient, it is obvious that the functions

$$\sigma\,\varphi(x),\ \sigma^4\,\varphi^{\mathrm{III}}(x),\ \sigma^5\,\varphi^{\mathrm{IV}}(x),\ \sigma^6\,\varphi^{\mathrm{V}}(x),\ .$$

are functions only of a single variable and hence may be conveniently tabulated with this variable as tabular argument.

I give a short table of the first three of these functions as well as of the probability integral at the end of this memoir.

In many instances the following abridged table will suffice for constructing a frequency curve (compare (5*)):

TABLE I.

$\frac{x-b}{\sigma}$	φ_0	φ_3	φ_4
− 3.0	+ 0.004	+ 0.080	+ 0.133
− 2.5	+ 0.018	+ 0.142	+ 0.030
− 2.0	+ 0.054	+ 0.108	− 0.270
− 1.5	+ 0.130	− 0.146	− 0.704
− 1.0	+ 0.242	− 0.484	− 0.484
− 0.5	+ 0.352	− 0.484	+ 0.550
0.0	+ 0.399	0.000	+ 1.197
+ 0.5	+ 0.352	+ 0.484	+ 0.550
+ 1.0	+ 0.242	+ 0.484	− 0.484
+ 1.5	+ 0.130	+ 0.146	− 0.704
+ 2.0	+ 0.054	− 0.108	− 0.270
+ 2.5	+ 0.018	− 0.142	+ 0.080
+ 3.0	+ 0.004	− 0.080	+ 0.133

The comparison between the observed and the calculated values of the frequency cannot be performed *directly* with the help of this table. For this purpose it is necessary to make use of the fuller tables at the end of this memoir. The frequency curve may, however, be constructed with the help of the above abridged table and compared with the empirical frequency curve. Compare the examples 1 and 4 beneath.

We write the series (1) in the form

(5) $$\sigma F(x) = \mu_0 [\sigma \varphi(x) + \beta_3 \sigma^4 \varphi^{III}(x) + \beta_4 \sigma^5 \varphi^{IV}(x) + \ldots]$$

or

(5*) $$\sigma F(x) = \mu_0 [\varphi_0(x) + \beta_3 \varphi_3(x) + \beta_4 \varphi_4(x) + \ldots],$$

where

$$\beta_3 = \frac{A_3}{A_0 \sigma^3};$$

$$\beta_4 = \frac{A_4}{A_0 \sigma^4};$$

and generally

$$\beta_s = \frac{A_s}{A_0 \sigma^s} = \frac{A_s}{\mu_0 \sigma^s}.$$

Using the abbreviation

(6) $$\nu_s = \mu_s : \mu_0$$

we obtain the following simple formulæ for the calculation of the coefficients β_3, β_4, .

(7) $$\begin{aligned} \underline{|3}\, \beta_3 &= - \nu_3 : \sigma^3, \\ \underline{|4}\, \beta_4 &= \nu_4 : \sigma^4 - 3, \\ \underline{|5}\, \beta_5 &= - \nu_s : \sigma_5 + 10\, \nu_3 : \sigma^3, \\ \underline{|6}\, \beta_6 &= \nu_6 : \sigma^6 - 15\, \nu_4 : \sigma^4 + 15, \end{aligned}$$

The functions $\varphi^s(x)$ are even functions of $x-b$, if s is an even number, and change the sign with $x-b$ if s is odd. Hence we find that the functions $\varphi^{III}(x)$, $\varphi^{V}(x), \ldots$ are liable to give to the frequency curve an unsymmetrical form, which is not the case with $\varphi^{IV}(x)$, $\varphi^{VI}(x)$, a. s. o. We find from the diagrams numbered 1, 2, 3, 4, 5 some instances of the influence of the first two terms on the form of the frequency curve.

Fig. 1 is the usual *normal-curve.* Figures 2 and 3 show the effect of different values of β_3 on the frequency curve. It is here supposed that β_4 and all other coefficients in (5) vanish. For great values of $x-b$ we here obtain negative values of the frequency, which is not possible in reality. The neglected terms of higher order must compensate those negative values. If β_5 and all following coefficients are small, it is convenient to choose β_3 as a measure of the skewness or dissymmetry of the curve. We hence will call β_3 *the coefficient of dissymmetry* (or skewness) of the frequency curve [1]). From the illustrations we may conclude, that a dissymmetry corresponding to the value $\beta_3 = 0.5$ must be considered as rather high, the frequency curve being then far different from the normal curve. It is to be expected, that in practice the value of β_3 will seldom exceed 0.5. The following coefficients in the series may however allow higher values of β_3 to occur.

The effect of the term $\beta_4 \sigma^5 \varphi^{IV}(x)$ may be shown from fig. 4 and 5, in which the normal curve is indicated by a dotted line.

For $\beta_4 = +0.1$ we obtain a curve similar to the normal curve, but it is directly observable from the figure that the number of individuals between $x - b = -\sigma$ and $x - b = +\sigma$ is greater when the frequency curve is characterized by $\beta_4 = +0.1$ than for $\beta_4 = 0$, when we have a normal distribution. The contrary takes place when $\beta_4 = -0.1$, or generally when β_4 has a negative value. We may conveniently, using an analogous nomenclature proposed by Pearson (Math. Contrib. I 1894), call β_4 *the excess of the frequency curve.*

In the simplest cases — and also the most usual ones — the coefficients β_3 and β_4 are sufficient to characterizise the frequency curves, naturally together with the *mean* (b), the *standard deviation* (σ) and the coefficient A_0 (μ_0), which latter equals the *area* of the frequency curve.

The equation (1) of the frequency curve being found it is easy to calculate the values of the *mode* and the *median*, which are sometimes used. For the *mode,* which corresponds to the maximum value of the frequency, we obtain the equation

$$0 = F'(x) = A_0 \varphi'(x) + A_3 \varphi^{IV}(x) + A_4 \varphi^{V}(x) + \ldots$$

If A_3 and A_4 are small quantities, as is here supposed, the value of $x - b$ satisfying this equation must be small. We obtain the following equation for the coordinate — x_1 — of the mode

$$(8^{**}) \qquad 0 = -z_1 + \beta_3 [3 - 6 z_1^2 + z_1^4] + \beta_4 [-15 z_1 + 10 z_1^3 - z_1^5] + \\ + \beta_5 [-15 + 30 z_1^2 - 15 z_1^4 + z_1^6] + \ldots,$$

where

$$z_1 = \frac{x_1 - b}{\sigma}.$$

Retaining only the terms of lowest order, we hence obtain

$$(8) \qquad z_1 = \frac{3 \beta_3}{1 + 15 \beta_4}$$

or, if β_4 be neglected,

$$(8^*) \qquad x_1 = b + 3 \sigma \beta_3 .$$

A more accurate value is easily obtained from the above equation (8**). The formula (8) may suffice for a general discussion of the position of the mode

[1]) I will also, for the sake of brevity, call β_3 the skewness of the frequency curve.

in relation to the mean. If the excess of the curve is small, it will be allowable to use the formula (8*).

As to the coordinate x_2 of the *median*, it may obtained in the following manner.

The median is defined in such a manner that the number of individuals between negative infinity and the median (x_2) is equal to the remaining number of individuals between x_2 and positive infinity. Hence the ordinate corresponding to $x = x_2$ divides the frequency curve into two equal parts.

We hence have

$$\int_{-\infty}^{x_2} F(x)\,dx - \int_{x_2}^{+\infty} F(x)\,dx = 0,$$

or, if the expression (1) for $F(x)$ is introduced,

$$(9^*) \qquad 0 = A_0 \int_{-\infty}^{x_2} \varphi(x)\,dx - A_0 \int_{x_2}^{+\infty} \varphi(x)\,dx + 2A_3\,\varphi^{II}(x_2) + 2A_4\,\varphi^{III}(x_2) + \ldots$$

For solving this equation we assume that A_3 and A_4, and in a still higher degree A_5 and the following coefficients, are small quantities. As

$$\int_{-\infty}^{b} \varphi(x)\,dx = \int_{b}^{\infty} \varphi(x)\,dx$$

it is therefore necessary, that x_2 has a value little different from b. We put

$$x_2 = b + \sigma z_2$$

and consider z_2 as a small quantity.

For developing (9*) in powers of z_2, we observe, that

$$\int_{-\infty}^{x_2} \varphi(x)\,dx = \int_{-\infty}^{b} \varphi(x)\,dx + \int_{b}^{x_2} \varphi(x)\,dx$$

$$= \frac{1}{2} + \int_{b}^{x_2} \varphi(x)\,dx,$$

and also

$$\int_{x_2}^{\infty} \varphi(x)\,dx = \frac{1}{2} - \int_{b}^{x_2} \varphi(x)\,dx,$$

so that

$$\int_{-\infty}^{x_2} \varphi(x)\,dx - \int_{x_2}^{+\infty} \varphi(x)\,dx = 2\int_{b}^{x_2} \varphi(x)\,dx$$

$$= 2\,\sigma \int_{0}^{z_2} \varphi(b + \sigma z)\,dz.$$

According to the value of $\varphi(x)$ we find that

$$\sigma\varphi(b+\sigma z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}},$$

and, developing this expression into powers of z and integrating, we thus finally find the following equation for determining z_2:

$$(9^{**}) \qquad 0 = z_2 - \frac{z_2^3}{6} + \ldots + \beta_3\left(-1 + \frac{3}{2} z_2^2 + \ldots\right) + \beta_4\left(3z_2 - \frac{5}{2} z_2^3 + \ldots\right).$$

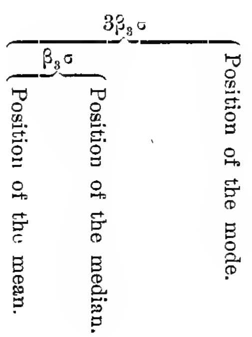
Neglecting β_5, β_6, ..., and terms of the third order we obtain

$$z_2 = \frac{\beta_3}{1+3\beta_4}$$

and hence we have

$$(9) \qquad x_2 = b + \frac{\sigma\beta_3}{1+3\beta_4}$$

For $\beta_3 = 0$ (β_5 and the higher coefficients being neglected) the mean, the mode and the median coincide. For frequency curves with small excess (for others we cannot conclude anything definitely from these formulæ) *the median is situated between the mean and the mode.*



The relative position of the mean, the median and the mode is first given by Pearson, who has derived it from his theory of frequency curves. For curves with a sensible *excess* the order of these points may possibly be different.

III. Numerical determination of the parameters of a frequency curve.

The calculation of the coefficients β_3, β_4, ... according to the formulæ (7) is a fairly simple affair, when the moments of the frequency curves are known. As the calculation of these moments has been thoroughly discussed by Pearson and his disciples, it would not be necessary to expend many words on this matter, were it not that some special points here deserve a closer examination. It ought to be demonstrated that the formula (1) is actually suitable to represent frequency curves, that is, that the number of coefficients in the series necessary for obtaining a practically sufficient representation is rather small. It will be shown that for most purposes it suffices to know the coefficients β_3 and β_4. When the series of observations on which the frequency curve is based is very numerous, it may be desirable to know the values of β_5 and β_6 also. This naturally is also the case, if the curve of frequency differs much from the normal curve.

As to the calculation of the moments of the curve I refer to the researches of PEARSON and SHEPPARD (Proc. Lond. Math. Soc. Vol. XXIX). The methods for obtaining the numerical values of the moments are clearly summarised by DAVENPORT (»Statistical Methods» P. 19 ff.). In a certain point it will be necessary to complete the numerical methods used by these authors, namely in respect to the checking of the numerical results. It must be considered as a rather laborious and imperfect method to check numerical work through double calculation or »calculation in pairs», as is recommended by the last named author. A scheme for numerical calculus must be so arranged, that errors may be detected by the computor himself, and such arrangements are generally easy to perform. In the first example I have carried out the control *in extenso.*

I bring here together the formulæ necessary for the calculation of the moments and of the coefficients of skewness and excess (β_3 and β_4).

(a) $$\mu_s' = \Sigma x^s F(x). \qquad (s = 0,\ 1,\ 2,\ 3,\ 4).$$

(b) $$\nu_s' = \mu_s' : \mu_0'.$$

Control:

(c) $$\Sigma (x - 1)^4 F(x) = \mu_4' - 4\mu_3' + 6\mu_2' - 4\mu_1' + \mu_0'.$$

or

(d) $$\Sigma (x + 1)^4 F(x) = \mu_4' + 4\mu_3' + 6\mu_2' + 4\mu_1' + \mu_0'.$$

(e) $$b = \nu_1',$$

(f) $$\sigma^2 = \nu_2 = \nu_2' - b^2,$$

(g) $$\nu_3 = \nu_3' - 3b\nu_2' + 2b^3,$$

(h) $$\nu_4 = \nu_4' - 4b\nu_3' + 6b^2\nu_2' - 3b^4.$$

Control:

(i) $$\nu_4' = \nu_4 + 4b\nu_3 + 6b^2\nu_2 + b^4.$$

(j) $$\beta_3 = - \nu_3 : 6\sigma^3,$$

(k) $$\beta_4 = \tfrac{1}{24} (\nu_4 : \sigma^4 - 3).$$

TABLE II. *Scheme for the calculation of frequency curves.*

$(x+1)^4$		x	$F(x)$	$xF(x)$	$x^2F(x)$	$x^3F(x)$	$x^4F(x)$	$(x+1)^4F(x)$	Control.
2401		-8							
1296		-7							
625		-6							
256		-5							μ_4'
81		-4							$4\mu_3'$
16		-3							$6\mu_2'$
1		-2							$4\mu_1'$
0		-1							μ_0'
		Σ_1							Σ_3
1		0							
16		$+1$							
81		$+2$							
256		$+3$							
625		$+4$							
1296		$+5$							
2401		$+6$							
4096		$+7$							
6561		$+8$							
		Σ_2							
		μ_s'							$=\Sigma_3$
		ν_s'							
				b	ν_2'	ν_3'	ν_4'		
		b		$b\nu_2'$		$b\nu_3'$		$b\nu_3$	
		b^2		$b^2\nu_2'$				$b^2\nu_2$	
		b^3							
		b^4							
		ν_2'		ν_3'		ν_4'		ν_4	
		$-b^2$		$-3b\nu_2'$		$-4b\nu_3'$		$4b\nu_3$	
		$\nu_2=\sigma^2$		$2b^3$		$6b^2\nu_2'$		$6b^2\nu_2$	
		σ		ν_3		$-3b^4$		b_4	
		σ^3		$\nu_3:\sigma^3$		ν_4		Σ_4	$=\nu_4'$
		σ^4		β_3		$\frac{\nu_4}{\sigma^4}-3$			
						β_4			

The last part of the calculus, (j) *and* (k) — by which β_3 and β_4 are obtained — *as well as* (b) *must be controlled through double calculation.*

A complete scheme for the calculation of a frequency curve according to the above formulæ is given on the preceding page.

When a certain statistical material in respect to a »collective object» is to be discussed, the first thing is to arrange this material into classes, all with the same extension (range) as to the character in question. The class range is taken as unity of the abscissæ. By inspection a class in the neighbourhood of the mean is chosen and considered as the origin from which the x-coordinates are reckoned. The two classes on both sides of that class, that is numerated with 0, get the number $+1$ and -1, and so on. The moments are calculated according to the equations (a)—(h). It is not necessary to take into account the corrections given by Pearson and Sheppard, if the class range is sufficiently small and coefficients of higher order than β_4 are not taken into consideration. As a rule it may be advisable *to take the class range smaller than the standard deviation,* the approximate value of which is easily found from the frequency table ($\frac{2}{3}$ of the material being included between the limits $b+\sigma$ and $b-\sigma$).

The corrected formulæ for the moments given by Sheppard are:

$$\sigma^2 = (\nu_2) = \nu_2' - b^2 - \tfrac{1}{12} = \nu_2 - \tfrac{1}{12}$$
$$(\nu_3) = \nu_3' - 3b\nu_2' + 2b^3 = \nu_3$$
$$(\nu_4) = \nu_4' - 4b\nu_3' + 6b^2\nu_2' - 3b^4 - \tfrac{1}{2}\sigma^2 - \tfrac{1}{80}$$
$$= \nu_4 - \tfrac{1}{2}\sigma^2 - \tfrac{1}{80},$$

where (ν_2), (ν_3) and (ν_4) design the corrected values of the moments (strictly the moments divided with μ_0).

1st Example. For illustrating the above general theory I begin with a frequency curve discussed by Davenport, belonging to the type I of Pearson [1]).

Distribution of frequency of glands in the right fore leg of 2000 female swine.

Number of glands	0	1	2	3	4	5	6	7	8	9	10
Frequency	15	209	365	482	414	277	134	72	22	8	2

We choose 4 glands as the provisional origin of the x-coordinates. The calculation scheme will then assume the following form.

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[1]) The frequency curve discussed in this example belongs, strictly spoken, to the type *B*, the curve obviously being limited in one direction. It may, however, be used as an example of such curves as, though belonging to the second type, may be conveniently represented through the formulæ of type *A*. If notable differences occur at the limited end of the curve between the observed and the calculated values, it will be necessary to use a curve of type *B*. I have treated the same curve as a *B*-curve beneath.

As to the controls it is to be remarked that the control (c), being a mere transposition of the terms, *must give full agreement between the two results to the last cipher.* As to the control of the second part of the calculus, through (i), a difference between the first and the second value of ν_4' may amount to some units of the last cipher. The difference in the example is 0.008, and hence rather great, and is probably caused by the neglected decimals in b^2 There is, however, no reason to make the calculation with more decimals.

All multiplications and divisions (partially also the additions) are performed with the aid of a calculating machine (I use for the present a machine of Odhners construction).

The five parameters hence have the following values:

$$\mu_0 = 2000,$$
$$b = -0.499,$$
$$\sigma = +1.681,$$
$$\beta_3 = -0.0848,$$
$$\beta_4 = +0.0046.$$

For comparing the observed values of the frequency with the theory we must calculate the values of φ_0, φ_3, φ_4 corresponding to the different classes. From tables B, C, D at the end of this memoir we obtain the values

n	0	1	2	3	4	5
x	− 3.5	− 2.5	− 1.5	− 0.5	+ 0.5	+ 1.5
$(x-b):\sigma$	− 2.07	− 1.49	− 0.893	− 0.296	+ 0.296	+ 0.893
φ_0	+ 0.047	+ 0.131	+ 0.268	+ 0.382	+ 0.382	+ 0.268
φ_3	− 0.124	− 0.153	− 0.527	− 0.329	+ 0.329	+ 0.527
φ_4	− 0.204	− 0.712	− 0.308	+ 0.947	+ 0.947	− 0.308

n	6	7	8	9	10
x	+ 2.5	+ 3.5	+ 4.5	+ 5.5	+ 6.5
$(x-b):\sigma$	+ 1.486	+ 2.07	+ 2.68	+ 3.27	+ 3.86
φ_0	+ 0.131	+ 0.047	+ 0.011	+ 0.002	+ 0.000
φ_3	+ 0.153	− 0.124	− 0.123	− 0.048	− 0.011
φ_4	− 0.712	− 0.204	+ 0.126	+ 0.102	+ 0.031

We hence derive the following values of the calculated frequencies compared with the observed ones.

Number of glands (n)	0	1	2	3	4	5	6	7	8	9	10
Observed frequency	15	209	365	482	414	277	134	72	22	8	2
Calculated ,,	43	166	369	494	427	264	136	67	26	8	1

The results are illustrated by fig. 6 on plate II.

TABLE III. *Distribution of frequency of glands in the right fore leg of 2000 female swine.*

n = number of glands.

Control.

$(x+1)^4$	n	x	$F(x)$	$xF(x)$	$x^2F(x)$	$x^3F(x)$	$x^4F(x)$	$(x+1)^4F(x)$	
2401		−8							
1296		−7							
625		−6							
256		−5							μ_4' + 48568
81	0	−4	15	− 60	+ 240	− 960	+ 3840	1215	$4\mu_3'$ − 15488
16	1	−3	209	− 627	+ 1881	− 5643	+ 16929	3344	$6\mu_2'$ + 36888
1	2	−2	365	− 730	+ 1460	− 2920	+ 5840	365	$4\mu_1'$ − 3992
0	3	−1	482	− 482	+ 482	− 482	+ 482	0	μ_0' + 2000
		Σ_1	1071	− 1899	+ 4063	− 10005	+ 27091	4924	Σ_3 + 67976
1	4	0	414	0	0	0	0	414	
16	5	+1	277	+ 277	+ 277	+ 277	+ 277	4432	
81	6	+2	134	+ 268	+ 536	+ 1072	+ 2144	10854	
256	7	+3	72	+ 216	+ 648	+ 1944	+ 5832	18432	
625	8	+4	22	+ 88	+ 352	+ 1408	+ 5632	13750	
1296	9	+5	8	+ 40	+ 200	+ 1000	+ 5000	10368	
2401	10	+6	2	+ 12	+ 72	+ 432	+ 2592	4802	
4096		+7							
6561		+8							
		Σ_2	929	+ 901	+ 2085	+ 6133	+ 21477	63052	
		μ_s'	2000	− 998	+ 6148	− 3872	+ 48568	67976	$= \Sigma_3$
		ν_s'		− 0.499	+ 3.074	− 1.936	+ 24.284		
				b	ν_2'	ν_3'	ν_4'		

b − 0.499	$b\nu_2'$ − 1.534	$b\nu_3'$ + 0.966	$b\nu_3$ − 1.207
b^2 + 0.249	$b^2\nu_2'$ + 0.765		$b^2\nu_2$ + 0.703
b^3 − 0.124			
b^4 + 0.062			

ν_2' + 3.074	ν_3' − 1.936	ν_4' + 24.284	ν_4 + 24.824
$-b^2$ − 0.249	$-3b\nu_2'$ + 4.602	$-4b\nu_3'$ − 3.864	$4b\nu_3$ − 4.828
$\nu_2 = \sigma^2$ + 2.825	$2b^3$ − 0.248	$6b^2\nu_2'$ + 4.590	$6b^2\nu_2$ + 4.218
σ + 1.681	ν_3 + 2.418	$-3b^4$ − 0.186	b^4 + 0.062
σ^3 + 4.749	$\nu_3 : \sigma^3$ + 0.509	ν_4 + 24.824	Σ_4 + 24.276 $= \nu_4'$
σ^4 + 7.983	β_3 − 0.0848	$\frac{\nu_4}{\sigma^4} - 3$ + 0.110	
		β_4 + 0.0046	

The agreement is as perfect as can be wished. The difference for $n=0$ and $n=1$ will diminnish, if a curve of type B be used. I have not considered this necessary in this case, as the curve of type A also gives a very good agreement. In example 8 I have in addition given a comparison of the same material with a curve of type B.

In constructing the curve of frequency I have not directly used the above values of the frequency. It is namely useful and instructive to reproduce the different frequency curves all in the same scale. For this purpose the standard deviation σ is taken as unit for the abscissæ and the numbers expressing the frequency are all multiplied by $\sigma : \mu_0$. As we have

$$\frac{\sigma}{\mu_0} F(x) = \varphi_0(x) + \beta_3 \varphi_3 + \beta_4 \varphi_4 + \cdots$$

we thus obtain for all frequency curves with the same values of β_3 and β_4 identically the same form. The construction of the curves of frequency is very simple, if the table I is used. The abscissæ of the *observed* values are obtained by means of the expression

$$\frac{x-b}{\sigma},$$

where x denotes the value of the character in question referred to the *provisional* origin. The comparison between theory and observation may conveniently be made with the help of the curve.

For the position of the mean, mode and median we obtain the values:

Mode: $x = 3.075$,
Median: $x = 3.359$,
Mean: $x = 3.501$.

Second Example. *Distribution of frequency of stigmatic bands of 1001 samples of Papaver.*

All the flowers were gathered in the same garden in Arild (Skåne) and counted by me the 27 July 1905.

Number of bands	4	5	6	7	8	9	10	11	12
Frequency	3	8	68	257	344	236	70	14	1

An easy calculation gives us, taking the *provisional* origin at 8,

$$\mu_0 = 1001,$$
$$b = -0.007,$$
$$\sigma = +1.142,$$
$$\beta_3 = -0.0006,$$
$$\beta_4 = +0.0093.$$

The curve is nearly normal, with the mode, mean and median at 8, a standard deviation equal to 1.142 and a small positive excess. In fig. 7 the observed frequencies are compared with a normal-curve (without excess).

Third Example. *Distribution of frequency in the weight of brown beans.*

Johannsen has made a very important investigation [1]) into the weight and other qualities of brown beans (*Phaseolus vulgaris*), which he has studied in many generations. What is specially characteristic in his researches is the self-fertilisation of the plants used in his experiments, so that it is possible for him to study the effect of *heredity* in its purest form. From the material published by him I take out his results respecting the weight of the beans in the third generation (1902). All the beans here considered derive in direct line from 19 grandmother-beans (1900), each constituting a line distinct from the other ones.

We have here to do with *graduated variates* (Davenport) that are capable of assuming all possible values within certain limits. In the first 2 examples the x-coordinates that measure the character in question, could assume only integer values. The graduated variates must be taken together in *classes*. The *class range* I take as by Johannsen to 50 mg. The provisional origin is for all lines taken to 475 mg. Hence class 1 has a mean weight of 525 mg, class 2 of 575 mg and so forth. The numbers given by Johannsen for the weight of the beans are contained in the following table.

TABLE IV *Frequency table of brown beans* (Johannsen).

Middle of the class	125	175	225	275	325	375	425	475	525	575	625	675	725	775	825	875	Σ
Class	—7	—6	—5	—4	—3	—2	—1	0	+1	+2	+3	+4	+5	+6	+7	+8	—
Line A	—	—	—	—	—	2	5	9	14	21	22	24	23	17	6	2	145
,, B	—	—	—	1	6	19	32	66	88	100	90	50	19	1	3	—	475
,, C	—	—	—	—	—	5	14	50	76	58	44	29	5	1	—	—	282
,, D	—	—	—	5	2	9	21	38	68	77	62	22	3	—	—	—	307
,, E	—	—	—	4	1	12	29	62	65	57	19	6	—	—	—	—	255
,, F	—	—	—	2	8	21	46	74	46	28	14	1	1	—	—	—	241
,, G	—	—	3	9	28	51	111	174	101	44	6	—	1	5	—	—	533
,, H	—	—	1	6	20	60	106	114	75	33	3	—	—	—	—	—	418
,, J	—	1	2	14	38	104	172	179	140	53	9	—	—	—	—	—	712
,, K	—	—	1	2	6	31	55	55	28	6	4	—	—	—	—	—	188
,, L	—	—	1	5	15	37	88	76	33	13	4	1	—	—	—	—	273
,, M	—	—	4	9	26	56	82	76	32	9	1	—	—	—	—	—	295
,, N	1	3	11	22	29	72	120	69	23	5	2	—	—	—	—	—	357
,, O	4	4	5	19	69	69	44	5	—	—	—	—	—	—	—	—	219
,, P	—	—	—	3	1	18	35	27	13	3	4	2	—	—	—	—	106
,, Q	—	—	1	2	7	16	44	93	80	52	10	—	—	—	—	—	305
,, R	—	—	—	2	3	12	17	27	19	3	—	—	—	—	—	—	83
,, S	—	—	1	2	3	8	27	47	37	30	4	—	—	—	—	—	159
,, T	—	—	—	—	1	6	20	37	39	30	8	—	—	—	—	—	141

Each line was treated according to the before given scheme. The corrections of Sheppard for the moments were not applied. The results were duly controlled. The values of the parameters for the different lines are contained in the following table.

[1]) »Ueber Erblichkeit in Populationen und in reinen Linien», Jena 1903 (Fischer).

TABLE V. *Parameters of frequency curves for pure lines of Phaseolus vulgaris.*

	μ_0	b	σ	β_3	β_4	m = Mean weight	σ in mg	$\sigma : m$
Line A	145	+3.338	2.177	+0.039	−0.022	642	109	0.170
„ B	475	+1.658	1.851	+0.032	−0.001	558	93	0.166
„ C	282	+1.585	1.524	−0.023	−0.015	554	76	0.138
„ D	307	+1.453	1.681	+0.120	+0.031	548	84	0.156
„ E	255	+0.737	1.497	−0.069	+0.021	512	75	0.146
„ T	141	+0.624	1.275	+0.036	−0.017	506	64	0.126
„ Q	305	+0.341	1.368	+0.013	+0.028	492	68	0.139
„ S	159	+0.277	1.427	+0.113	+0.038	489	71	0.146
„ F	241	+0.137	1.517	−0.010	+0.004	482	76	0.158
„ G	533	−0.200	1.562	−0.031	+0.082	465	78	0.167
„ H	418	−0.395	1.396	+0.033	−0.005	455	70	0.154
„ R	83	−0.398	1.308	+0.094	± 0.000	455	65	0.144
„ J	712	−0.411	1.477	+0.045	± 0.000	454	74	0.163
„ P	106	−0.443	1.493	−0.094	+0.044	453	75	0.165
„ K	188	−0.511	1.323	+0.002	+0.026	449	66	0.147
„ L	273	−0.576	1.383	−0.007	+0.026	446	69	0.155
„ M	295	−0.932	1.432	+0.048	+0.004	428	72	0.168
„ N	357	−1.344	1.560	+0.094	+0.032	408	78	0.191
„ O	219	−2.474	1.299	+0.166	+0.075	351	65	0.182

The lines are here ordered according to their mean weight, which varies between 642 mg and 351 mg. The *standard deviation* (σ) varies between 109 mg and 64 mg and seems to depend on the magnitude of the beans, being nearly proportional to their mean weight. This fact is shown by the last column, which gives the quotient between the standard deviation and the mean weight of the beans. Taking the mean of the numbers in the last column, we find that the standard deviation amounts to 15,7 % of the weight of the beans.

The frequency curves of most pure lines show a good agreement with the normal curve. Some pure lines, however, have a frequency curve with a notable *skewness*, as the lines D, S, R, P, N and O. The greatest value of the skewness occurs for the line O, where $\beta_3 = +0.166$. As to the *excess,* we find that a negative excess occurs rather seldom. The greatest positive excess occurs at the lines G and O, amounting at the most (in the line G) to $+0.082$. The form of the frequency curve of the line O, which has the greatest deviation from the normal curve, is shown in fig. 10. In fig. 8 and 9 I give the frequency curves of the lines A and G. The agreement between theory and observation is generally tolerably good, the most notable exception occurring in line O, where the number of beans with the mean weight 375 mg seems to be too small.

On the connexion between the values of the parameters and hereditary circumstances I have made some researches, till now with negative result. When it becomes possible to compare the results from many generations, it seems probable that such a study will show itself more fertile. For the present I will only point out the simple and instructive description of a frequency curve that is given through the coefficients β_3 and β_4. They give a most concentrated idea of the curve and allow one to calculate the theoretical frequency curve in the most simple manner.

Fourth Example. *Distribution of frequency in the cephalic index of 22505 Swedish recruits.*

In an important work »*Anthropologia suecica*» M. RETZIUS and FÜRST have studied the Swedish recruits in the years 1897 and 1898 in different respects of interest for statistical anthropology. From this work I take out the following numbers relating to the cephalic index (»Schädelindex») of 22505 Swedish recruits in the year 1897.

Cephalic index	65.5	67.5	69.5	71.5	73.5	75.5	77.5
Class	— 6	— 5	— 4	— 3	— 2	— 1	0
Frequency	12	87	510	1952	4346	6039	5050

Cephalic index	79.5	81.5	83.5	85.5	87.5	89.5
Class	+ 1	+ 2	+ 3	+ 4	+ 5	+ 6
Frequency	2822	1172	377	94	31	13

The class range is here equal to two integer cephalic indices, the above numbers being the sum of the frequencies relating to two consecutive cephalic indices in the table of RETZIUS and FÜRST. The provisional origin is taken at 77.5. The reckoning according to the given scheme gave the following values of the parameters:

$$\mu_0 = 22505,$$
$$b = -0.721,$$
$$\sigma = +1.544,$$
$$\beta_3 = -0.0404,$$
$$\beta_4 = +0.0151.$$

Hence the mean was equal to 76.058, and the standard deviation expressed in cephalic indices was 3.088.

The above results were obtained with the *uncorrected* values of the moments.

In fig. 11 is shown the graphical comparison between theory and observation. The agreement is very perfect as may be expected from such an extensive material.

It must be remarked that this treatment of the beautiful material relating to the cephalic index of Swedish recruits has a quite provisional character. In the above calculation 22 individuals with extreme index values have been excluded. Their retaining claims a fuller discussion of the problem than is for the moment possible for me to give.

Fifth Example. *Typhoid fever in Lund 1905.*

As a last example of frequency curves belonging to type A, I take a case from medical statistics, namely the typhoid fever in Lund in this year. The following numbers are taken from an official account on this fever, which appeared in »Lunds Dagblad» in September this year. As the numbers fluctuated rather much from one day to another, I have taken together the results for three consecutive days. Thus on the 7th, 8th and 9th August there occurred in all 11 cases of typhoid fever, on the 10th, 11th and 12th in all 24 cases a. s. o.

Date	Aug. 5,	8,	11,	14,	17,	20,	23,	26,	29,	Sept. 1,	4,	7.
Frequency	2,	11,	24,	49,	46,	32,	16,	23,	10,	5,	2,	0.5.
Class	−6,	−5,	−4,	−3,	−2,	−1,	0,	+1,	+2,	+3,	+4,	+5.

Hence the class range is equal to 3 days. The provisional origin was taken at the 23th August.

For the parameters of the theoretical frequency curve I obtained the values

$$\begin{aligned} \mu_0 &= 220.5, \\ b &= -1.658, \\ \sigma &= +2.058, \\ \beta_3 &= -0.0882, \\ \beta_4 &= -0.0047. \end{aligned}$$

The mean corresponds to the date Aug. 18.0, the standard deviation amounts to 6.17 days. The comparison between theory and observation is shown from fig. 12. The discrepancies are here rather great, as may be expected from such material. It is obviously connected with great difficulties to determine with some certainty the beginning of the disease in each individual case. Probably accuracy may be augmented if the attention of the physicians is directed to the importance of accurate statistical determinations.

Notwithstanding the imperfection of the material, we find that the theoretical frequency curve reproduces the general features of the curve indicated by the observations fairly well. The negative skewness implies that the increase in the number of infected persons is more rapid than the subsequent decrease after the maximum is reached. This is perhaps characteristic for all such fever maladies.

IV. Type B of frequency curves.

This type is expressed by means of the generating function

$$(10) \qquad \psi(x) = \frac{e^{-\lambda} \sin \pi x}{\pi} \left[\frac{1}{x} - \frac{\lambda}{\underline{|1}\,(x-1)} + \frac{\lambda^2}{\underline{|2}\,(x-2)} - \frac{\lambda^3}{\underline{|3}\,(x-3)} + \ldots \right].$$

We write $\psi_\lambda(x)$, for $\psi(x)$, if we want to indicate that a parameter λ occurs in $\psi(x)$. We find from (10) that $\psi(x)$ is a *whole transcendent function of* x, which hence is infinite for no finite value of x. For $x = -1, -2, -3, \ldots$ $\psi(x)$ vanishes. Considering $\psi_\lambda(x)$ as a function of λ, we also find that this function is a whole transcendent function of λ. I have given (»Meddelanden» N:o 26) for $\psi(x)$ also another form, as an integral, namely

$$(10^*) \qquad \psi(x) = \frac{e^{-\lambda}}{\pi} \int_0^{\pi} e^{\lambda \cos \omega} \cos [\lambda \sin \omega - x\omega]\, d\omega,$$

which may sometimes be preferable to the series (10). If r be a *positive integer*, we have

$$(11) \qquad \psi(r) = \frac{e^{-\lambda} \lambda^r}{\underline{|r}}.$$

In the following investigation we shall find, that, by suitably choosing the parameters c, ω and λ, a frequency curve *approximately* may be represented by means of the formula

$$F(x) = B_0 \psi_\lambda \left(\frac{x - c}{\omega}\right).$$

Hence the function $\psi_\lambda(x)$ will give for different values of λ the differents forms of the frequency curves of type B. In fig. 13 I have reproduced some of these forms, where it may be observed that only integer values of x are taken into consideration. We find that the frequency curves of type B for $x = c$ discontinuously breaks up and possesses a finite value, whereas for $x = \infty$ $\psi_\lambda(x)$ tends towards zero. With increasing λ the curves gradually approach the form of the curves of type A.

More generally we may write a frequency curve of the type B in the form

$$(12) \qquad F(x) = B_0\psi(x) + B_1\Delta\psi(x) + B_2\Delta^2\psi(x) + B_3\Delta^3\psi(x) + \ldots,$$

where (»Meddelanden» N:o 27) the coefficients have the following values

$$\begin{aligned}
B_0 &= \mu_0', \\
B_1 &= \lambda\mu_0' - \mu_1', \\
\underline{|2}\, B_2 &= \lambda^2\mu_0' - (2\lambda + 1)\,\mu_1' + \mu_2', \\
\underline{|3}\, B_3 &= \lambda^3\mu_0' - (3\lambda^2 + 3\lambda + 2)\,\mu_1' + 3\,(\lambda + 1)\,\mu_2' - \mu_3', \\
\underline{|4}\, B_4 &= \lambda^4\mu_0' - (4\lambda^3 + 6\lambda^2 + 8\lambda + 6)\,\mu_1' + (6\lambda^2 + 12\lambda + 11)\,\mu_2' \\
&\quad - (4\lambda + 6)\,\mu_3' + \mu_4', \\
&\quad \cdots\cdots
\end{aligned}$$

and μ_0', μ_1', μ_2', ... are defined by the formula

$$(12^*) \qquad \mu_s' = \sum_{-\infty}^{+\infty} x^s F(x). \qquad (s = 0,\ 1,\ 2,\ \ldots)$$

The parameter λ may be arbitrarily chosen. It is possible to introduce two new parameters, if we write instead of (12)

$$(13) \qquad F(x\omega + c) = B_0\psi(x) + B_1\Delta\psi(x) + B_2\Delta^2\psi(x) + B_3\Delta^3\psi(x) + \ldots$$

It is now

$$B_r = \sum_{x=-\infty}^{+\infty} T_r(x) F(x\omega + c)$$

$$= \sum T_r\left(\frac{y-c}{\omega}\right) F(y),$$

in which formula y must assume all values given by the relation

$$y = x\omega + c,$$

where $x = 0, \pm 1, \pm 2, \pm 3, \ldots$ *in inf.*

As to $T_r(x)$ we know that it is a polynome of degree r in x. If we write

$$T_r(x) = \delta_0^{(r)} x^r + \delta_1^{(r)} x^{r-1} + \ldots + \delta_{r-1}^{(r)} x + \delta_r^{(r)},$$

and observe that

$$\mu_s'' = \omega \sum (y-c)^s F(y) = \omega \sum_{r=-\infty}^{+\infty} (r\omega)^s F(c + r\omega)$$

is dependent on c, but independent of ω (if ω is rather small), we have

$$\omega^{r+1} B_r = \delta_0^{(r)} \mu_r'' + \delta_1^{(r)} \omega \mu_{r-1}'' + \ldots + \delta_{r-1}^{(r)} \omega^{r-1} \mu_1'' + \delta_r^{(r)} \omega^r \mu_0'',$$

so that the values of B_0, B_1, ... now are

$$\omega B_0 = \mu_0'',$$
$$\omega^2 B_1 = \lambda\omega\mu_0'' - \mu_1'',$$
$$(14) \quad \omega^3 \underline{|2}\, B_2 = \lambda^2\omega^2\mu_0'' - (2\lambda + 1)\,\omega\mu_1'' + \mu_2'',$$
$$\omega^4 \underline{|3}\, B_3 = \lambda^3\omega^3\mu_0'' - (3\lambda^2 + 3\lambda + 2)\,\omega^2\mu_1'' + 3(\lambda + 1)\,\omega\mu_2'' - \mu_3'',$$
$$\omega^5 \underline{|4}\, B_4 = \lambda^4\omega^4\mu_0'' - (4\lambda^3 + 6\lambda^2 + 8\lambda + 6)\,\omega^3\mu_1'' + (6\lambda^2 + 12\lambda + 11)\,\omega^2\mu_2'' - (4\lambda + 6)\,\omega\mu_3'' + \mu_4'',$$
$$\ldots\ldots$$

The frequency curves of the type B may be treated mathematically in different manners. In the general formula (13) ω, b and λ may be arbitrarily chosen. The greatest convergency is generally attained if these constants are determined in such a manner that $B_1 = B_2 = B_3 = 0$. It is, however, *not necessary* to choose the parameters in this manner. Sometimes it will be found convenient to give to λ, c or ω determinate values. We will treat some of these values.

1:o. *We put* $\omega = 1$ *and* $c = 0$.

It is now

$$(15) \qquad F(x) = B_0 \psi_\lambda(x) + B_1 \Delta\psi + B_2 \Delta^2\psi + B_3 \Delta^3\psi + \ldots$$

Dividing the expressions for B_1, B_2, B_3, ... by B_0, we obtain, if we put

$$(16^*) \qquad \mu_0' \nu_s' = \mu_s',$$

$$
(16)\qquad
\begin{aligned}
B_1 &= B_0\,(\lambda - \nu_1'),\\
\underline{|2}\, B_2 &= B_0\,(\lambda^2 - (2\lambda + 1)\,\nu_1' + \nu_2'),\\
\underline{|3}\, B_3 &= B_0\,(\lambda^3 - (3\lambda^2 + 3\lambda + 2)\,\nu_1' + 3\,(\lambda + 1)\,\nu_2' - \nu_3'),\\
\underline{|4}\, B_4 &= B_0\,(\lambda^4 - (4\lambda^3 + 6\lambda^2 + 8\lambda + 6)\,\nu_1' + (6\lambda^2 + 12\lambda + 11)\,\nu_2'\\
&\qquad - (4\lambda + 6)\,\nu_3' + \nu_4'),\\
&\quad \dots
\end{aligned}
$$

We give to λ such a value that the coefficient B_1 vanishes. We then have, putting $\nu_1' = b$,

$$
\begin{aligned}
B_1 &= 0,\\
\underline{|2}\, B_2 &= B_0\,(\nu_2' - b^2 - b),\\
\underline{|3}\, B_3 &= B_0\,(-2b^3 - 3b^2 - 2b + 3b\nu_2' + 3\nu_2' - \nu_3'),\\
\underline{|4}\, B_4 &= B_0\,(-3b^4 - 6b^3 - 8b^2 - 6b + (6b^2 + 12b + 11)\,\nu_2'\\
&\qquad - (4b + 6)\,\nu_3' + \nu_4'),\\
&\quad \dots
\end{aligned}
$$

We here introduce the moments about the mean that are defined by the equations

$$
(17^*)\qquad \mu_0\,\nu_s = \Sigma\,(x - b)^s\, F(x) \qquad (s = 0,\ 1,\ 2,\ .\ \ .),
$$

b being the coordinate of the mean, so that

$$
\begin{aligned}
\nu_2' &= \nu_2 + b^2,\\
\nu_3' &= \nu_3 + 3b\nu_2 + b^3,\\
\nu_4' &= \nu_4 + 4b\nu_3 + 6b^2\nu_2 + b^4,\\
&\quad \dots
\end{aligned}
$$

which relations are obvious, if we remember that the mean is determined in such a manner that the first moment about it vanishes.

The expressions for B_2, B_3 and B_4 now assume the simple form

$$
(17)\qquad
\begin{aligned}
\underline{|2}\, B_2 &= B_0\,(\nu_2 - b),\\
\underline{|3}\, B_3 &= B_0\,(-\nu_3 + 3\nu_2 - 2b),\\
\underline{|4}\, B_4 &= B_0\,(\nu_4 - 6\nu_3 - 6b\nu_2 + 11\nu_2 + 3b^2 - 6b),\\
&\quad \dots
\end{aligned}
$$

When the moments about the mean are known, the coefficients B_2, B_3, B_4 are easily obtained from (17), and we have

$$
(17^{**})\qquad F(x) = \mu_0\,\psi_\lambda(x) + B_2\,\Delta^2\psi + B_3\,\Delta^3\psi + B_4\,\Delta^4\psi + \dots,
$$

where now $\lambda = b = \nu_1'$.

2:o. *We put* $\omega = 1$, leaving c undetermined.

If we employ the parameters c and λ to make vanish the coefficients B_1 and B_2, we now have

$$B_0 = \mu_0,$$
$$c = b - \nu_2,$$
$$\lambda = \nu_2,$$

and it is

$$\underline{|3}\, B_3 = \mu_0 (\nu_2 - \nu_3),$$
$$\underline{|4}\, B_4 = \mu_0 (\nu_4 - 3\nu_2^2 - 6\nu_3 + 5\nu_2),$$

where it is supposed that

$$\Sigma\, (c + x)^s F(c + x) = \Sigma\, x^s F(x) = \mu_s'.$$

3:o. *We determine* λ, ω *and* c *in such a manner that* $B_1 = B_2 = B_3 = 0$. Multiplying (13) by 1, x, x^2 and x^3, we then obtain the equations

$$(18)\qquad \begin{aligned} \sum_{x=-\infty}^{+\infty} F(x\omega + c) &= B_0\, \Sigma\, \psi(x) = B_0, \\ \Sigma\, x F(x\omega + c) &= B_0\, \Sigma\, x\psi(x) = B_0 \lambda, \\ \Sigma\, x^2 F(x\omega + c) &= B_0\, \Sigma\, x^2\psi(x) = B_0 (\lambda^2 + \lambda), \\ \Sigma\, x^3 F(x\omega + c) &= B_0\, \Sigma\, x^3\psi(x) = B_0 (\lambda^3 + 3\lambda^2 + \lambda). \end{aligned}$$

These equations may be regarded as exact ones. For solving them in respect to B_0, ω, b, λ we must have recourse to approximations. Defining the moments μ_s' of the frequency curve about a provisional origin by the equation (12*), we suppose that

$$(19)\qquad \mu_s' = \sum_{x=-\infty}^{+\infty} (x\omega + c)^s\, \omega F(x\omega + c)$$

and hence — using this value of μ_s' — we have

$$\begin{aligned} \omega\, \Sigma\, F(x\omega + c) &= \mu_0', \\ \omega^2\, \Sigma\, xF(x\omega + c) &= \omega\, \Sigma\, (\omega x + c - c)\, F(x\omega + c) \\ &= \mu_1' - c\mu_0', \\ \omega^3\, \Sigma\, x^2 F(x\omega + c) &= \omega\, \Sigma\, (\omega x + c - c)^2 F(x\omega + c) \\ &= \mu_2' - 2c\mu_1' + c^2 \mu_0', \\ \omega^4\, \Sigma\, x^3 F(x\omega + c) &= \mu_3' - 3c\mu_2' + 3c^2 \mu_1' - c^3 \mu_0'. \end{aligned}$$

The above equations (16) then assume the form

$$\begin{aligned} \mu_0' &= \omega B_0, \\ \mu_1' - c\mu_0' &= \omega^2 B_0 \lambda, \\ \mu_2' - 2c\mu_1' + c^2 \mu_0' &= \omega^3 B_0 (\lambda^2 + \lambda), \\ \mu_3' - 3c\mu_2' + 3c^2 \mu_1' - c^3 \mu_0' &= \omega^4 B_0 (\lambda^3 + 3\lambda^2 + \lambda), \end{aligned}$$

or, putting

$$\mu_0 \nu_s' = \mu_s',$$

$$\nu_1' - c = \omega\lambda,$$
$$\nu_2' - 2c\nu_1' + c^2 = \omega^2(\lambda^2 + \lambda),$$
$$\nu_3' - 3c\nu_2' + 3c^2\nu_1' - c^3 = \omega^3(\lambda^3 + 3\lambda^2 + \lambda).$$

In these relations we introduce the moments about the mean, the coordinate of which relating to the provisional origin is called b. The above equations now assume the form

$$b - c = \omega\lambda,$$
$$\nu_2 + (b-c)^2 = \omega^2(\lambda^2 + \lambda),$$
$$\nu_3 + 3\nu_2(b-c) + (b-c)^3 = \omega^3(\lambda^3 + 3\lambda^2 + \lambda),$$

the solution of which is

(20) $$\begin{cases} \omega = \nu_3 : \nu_2, \\ \lambda = \nu_2^3 : \nu_3^2, \\ c = b - \nu_2^2 : \nu_3. \end{cases}$$

Finally we have

$$B_0 = \mu_0 : \omega.$$

Hence we find that the parameters are very easily calculated from the moments of the frequency curve.

We now have

(21) $$F(x\omega + c) = B_0\psi(x) + B_4\Delta^4\psi(x) + B_5\Delta^5\psi(x) + \ldots,$$

where generally it is superfluous to know the values of B_4 and B_5.

Putting

$$y = x\omega + c$$

we may write this equation in the form

(21*) $$F(y) = B_0\psi\left(\frac{y-c}{\omega}\right) + B_4\Delta^4\psi + B_5\Delta^5\psi + \ldots$$

In applying this formula it is necessary to define $\psi(x)$ by the general formula (10), the argument being generally not an integer. Unfortunately there does not yet exist a table of the function $\psi(x)$ for such values of the argument as are not integer.

As a control we derive from (20) the relation:

(22) $$\omega^2\lambda = \sigma^2,$$

where σ signifies the standard deviation.

For the coefficient B_4 I have obtained the value

(23) $$B_4 = \frac{B_0}{24\,\omega^4}\left(\nu_4 - 3\nu_2^2 - \frac{\nu_3^2}{\nu_2}\right).$$

4:o. *The quantities λ and ω are so determined that $B_1 = B_2 = 0$, whereas c is chosen arbitrarily.*

The method 3:o may seem to be the best one, but has the inconvenience of giving to ω very small values and to λ very large ones, when ν_3 is vanishing. Hence it is not applicable when the curve differs little from the normal-form. The following method seems to have a general applicability and has also the advantage of a certain similarity with the process used for the curves of type *A*.

We begin with choosing a determinate value for the quantity c. *In many cases it will be found convenient to identify c with the abscissa of the discontinuous end of the frequency curve.*

When the value of the quantity c is determined (and it must be borne in mind that this determination is to a certain degree arbitrary) we dispose of λ and ω in such a manner that the coefficients B_1 and B_2 vanish. According to (14) we thus get the equations of condition

$$\begin{aligned} 0 &= \lambda\omega\mu_0'' - \mu_1'', \\ 0 &= \lambda^2\omega^2\mu_0'' - (2\lambda + 1)\,\omega\mu_1'' + \mu_2''. \end{aligned} \tag{24}$$

For solving these equations we observe that the moments μ_s'', which are taken about the point c, may be expressed through the moments μ_s about the mean. We have indeed *approximately*:

$$\mu_s'' = \mu_s + \binom{s}{1}(b - c)\,\mu_{s-1} + \binom{s}{2}(b - c)^2\mu_{s-2} + \ldots$$

As $\mu_1 = 0$ we thus obtain

$$\begin{aligned} \mu_0'' &\doteq \mu_0, \\ \mu_1'' &= (b - c)\,\mu_0, \\ \mu_2'' &= \mu_2 + (b - c)^2\mu_0, \\ \mu_3'' &= \mu_3 + 3\,(b - c)\,\mu_2 + (b - c)^3\mu_0, \\ \mu_4'' &= \mu_4 + 4\,(b - c)\,\mu_3 + 6\,(b - c)^2\mu_2 + (b - c)^4\mu_0, \\ &\ldots \end{aligned}$$

Substituting these values in (24) we get the following values of λ and ω:

$$\left\{ \begin{aligned} \lambda &= \frac{(b - c)^2}{\sigma^2}, \\ \omega &= \frac{\sigma^2}{b - c}, \end{aligned} \right. \tag{25}$$

where $\sigma^2 (= \nu_2)$ signifies the standard deviation.

As to B_3 and B_4 they now assume the values:

$$\begin{aligned} \omega^3\,\underline{|3}\,B_3 &= B_0\,[\omega\nu_2 - \nu_3], \\ \omega^4\,\underline{|4}\,B_4 &= B_0\,[\nu_4 - 3\nu_2^2 + 5\omega^2\nu_2 - 6\omega\nu_3]. \end{aligned} \tag{26}$$

Hence we may write the frequency curve in the form

$$F(x\omega + c) = \frac{\mu_0}{\omega}\,[\psi(x) + \gamma_3\Delta^3\psi + \gamma_4\Delta^4\psi + \ldots], \tag{27}$$

where

(28)
$$\omega^3 \underline{|3}\, \gamma_3 = \omega\nu_2 - \nu_3,$$
$$\omega^4 \underline{|4}\, \gamma_4 = \nu_4 - 3\nu_2^2 + 5\omega^2\nu_2 - 6\omega\nu_3.$$

These expressions we may also write in the following form

$$\gamma_3 = \frac{1}{\underline{|3}}\lambda - \frac{\nu_3}{\underline{|3}\,\sigma^3}\lambda^{\frac{3}{2}},$$

$$\gamma_4 = \frac{\nu_4 - 3\nu_2^2}{\underline{|4}\,\sigma^4}\lambda^2 + \frac{5}{24}\lambda - \frac{6}{\underline{|4}}\frac{\nu_3}{\sigma^3}\lambda^{\frac{3}{2}},$$

or, introducing the coefficients β_3 and β_4 belonging to the curves of type A,

(29)
$$\begin{cases} \gamma_3 = \dfrac{1}{\underline{|3}}\lambda + \beta_3\lambda^{\frac{3}{2}}, \\ \gamma_4 = \dfrac{5}{24}\lambda + \dfrac{3}{2}\beta_3\lambda^{\frac{3}{2}} + \beta_4\lambda^2, \end{cases}$$

in which form the calculation of the coefficients for the curves of type B is easily performed.

For graphical construction it will be suitable to write the equation of the frequency curve in the form

(30)
$$\frac{\sigma}{\mu_0} F(x\omega + c) = \sqrt{\lambda}\,[\psi(x) + \gamma_3\Delta^3\psi + \gamma_4\Delta^4\psi + \ldots].$$

The formulæ (25), (27) and (28) contain all that is necessary for the calculation of the curves of type B. The numerical operation is substantially the same for the curves of both types. The calculation of μ_0, ν_2, ν_3, ν_4, σ, b, β_3, β_4 is executed according to the scheme II. Then λ and ω are calculated with the help of (25), and γ_3 and γ_4 by the formulæ (29). The graphical construction and the comparison with the observation is performed with the help of (30). As for the present the values of the function $\psi(x)$ are tabulated only for integer values of the argument the comparison between observation and theory must take place in graphical manner. The values of $\psi(x)$ for integer values of x are given according to Bortkewitsch, in tab. E.

It is supposed in these investigations on the curves of type B, that

(31)
$$\omega\,\Sigma\,(x\omega + c)^s F(x\omega + c) = \Sigma\, x^s F(x),$$

where x takes all integer values between $-\infty$ and $+\infty$. In many cases, however, this relation must be regarded only as a rough approximation. It is necessary to calculate the corrections to this formula and the resulting corrections to the expressions of the parameters of the frequency curve. For want of time I have not at present opportunity to work out these formulæ (the corrections of Sheppard are not here sufficient), but will confine myself to making an observation on a single point.

We suppose c to be the abscissa of the discontinuous end of the frequency curve. It is then $0 = F(c - \omega) = F(c - 2\omega) = \ldots.$ Put $s = 0$.

The area — Y — between the frequency curve and the line of the abscissæ may approximately be written

$$Y = \omega\,[\tfrac{1}{2}F(c) + F(c+\omega) + F(c+2\omega) + \ldots]$$

or also

$$Y = \tfrac{1}{2}F(c) + F(c+1) + F(c+2) + \ldots$$

Using the abbreviation

$$\mu_0 = F(c) + F(c+1) + F(c+2) + \ldots,$$

which is adequate when *integral* variates are concerned, we thus have

$$\omega\,\Sigma\,F(x\omega + c) = \mu_0 + \tfrac{1}{2}F(c)\,(\omega - 1),$$

whereas in the preceding investigation the term multiplied by $F(c)$ was omitted.

Using only this correction the equations of condition in case 4:o take the form

$$\text{(32)}\qquad \begin{aligned} \mu_0 + \tfrac{1}{2}F(c)\,(\omega - 1) &= \omega B_0, \\ \mu_1' &= \omega^2 B_0 \lambda, \\ \mu_2' &= \omega^3 B_0 (\lambda^2 + \lambda), \end{aligned}$$

which equations may be exactly solved.

Putting

$$h = \frac{F(c)}{\mu_0}$$

we obtain ω from the equation

$$\text{(33)}\qquad \omega^2 + 2\omega\left[\frac{1}{h} - \frac{1}{2} - \frac{1}{2}b - \frac{\sigma^2}{2b}\right] = \frac{\sigma^2}{b}\left(\frac{2}{h} - 1\right) - b,$$

then λ from

$$\text{(33*)}\qquad \omega\lambda = \frac{b}{1 + \frac{1}{2}h\,(\omega - 1)},$$

and B_0 from

$$\text{(33**)}\qquad B_0 = \frac{\mu_0}{\omega}\,(1 + \tfrac{1}{2}h\,(\omega - 1)).$$

When $F(c)$ is small, we may conveniently develop the solution of (32) into powers of $F(c)$. In the first approximation we then obtain the solution (25), which solution will suffice when $F(c)$ is very small. Compare in this respect the problem 8 below.

V. Numerical applications.

I will apply the above general theory to some examples.

Sixth Example. *Number of petals of Ranunculus bulbosus.*

The following numbers are given by HUGO DE VRIES and treated by PEARSON (»Contributions» 1895).

Class	0	1	2	3	4	5
Number of petals	5	6	7	8	9	10
Frequency	133	55	23	7	2	2

We will represent this numbers by means of a frequency curve of form B, putting $c = 0$, $\omega = 1$, that is using method 1:o above.

We find by a comparison between these numbers and fig. 13 that λ has a value smaller than unity. Placing the provisional origin at the class 0, representing the individuals with 5 petals, we find

$$\mu_0' = 222,$$
$$\mu_1' = 140,$$
$$\mu_2' = 292.$$

More moments it is not necessary to calculate in this case. From these numbers we obtain

$$\nu_1' = +\,0.631$$
$$\nu_2' = +\,1.314 \qquad \nu_2 = +\,0.916$$

As the value of ν_1' seems to be not very distant from the value of λ, it will be advantageous to use method 1:o and, according to the formulæ (16) and (17), we then obtain

$$\lambda = \nu_1' = +\,0.631,$$
$$B_2 = +\,31.5,$$

so that

$$F(x) = 222\,\psi(x) + 31.5\,\Delta^2\psi.$$

From the table of BORTKEWITSCH we obtain through interpolation the following values of $\psi_\lambda(x)$ corresponding to $\lambda = 0.631$. The values of $\Delta\psi$ and $\Delta^2\psi$ are obtained by taking the differences:

x	$\psi(x)$	$\Delta\psi$	$\Delta^2\psi$
0	+ 0.532	+ 0.532	+ 0.532
1	+ 0.336	— 0.196	— 0.728
2	+ 0.106	— 0.230	— 0.034
3	+ 0.022	— 0.084	+ 0.146
4	+ 0.003	— 0.018	+ 0.066
5	+ 0.000	— 0.003	+ 0.015

It is to be observed that $\psi(-1) = \psi(-2) = 0$, as follows directly from the formula (10).

We now derive the following values of $F(x)$:

x	$222\,\psi(x)$	$31.5\,\Delta^2\psi$	$F(x)$	
			Calc.	Obs.
0	+ 118.1	+ 16.8	134.9	133
1	+ 74.5	− 22.9	51.6	55
2	+ 23.6	− 1.1	22.5	23
3	+ 4.9	+ 4.6	9.5	7
4	+ 0.8	+ 2.1	2.9	2
5	+ 0.1	+ 0.5	0.6	2

The agreement is as complete as can be wished. The effect of the second differences is clearly pronounced and is rather considerable for $x = 0$ and $x = 1$.

A great advantage with this method is, that it is only necessary to calculate directly the values of $\psi(x)$, whereas the values of $\Delta\psi$, $\Delta^2\psi$, $\Delta^3\psi$, . . . are obtained through the numerical differences of $\psi(x)$, which are easily obtained.

Seventh Example. *Variation in the failing percentage of barley.*

In his lectures on the theory of heredity W. Johannsen has given some instances of frequency curves that belong to type *B*. From these I choose the following one relating to the failing of the grains of common barley.

Mean procent of failing	2.5	7.5	12.5	17.5	22.5	27.5	32.5	37.5	42.5	47.5	52.5
Class	0	1	2	3	4	5	6	7	8	9	10
Frequency	53	131	180	170	111	50	22	22	7	2	1

From these numbers we obtain the following values of the moments

$$\mu_0 = 749, \qquad \nu_2 + 3.063,$$
$$b = +2.757, \qquad \nu_3 + 4.250,$$
$$\nu_4 + 35.464.$$

Calculating now ω, λ and c according to the third method above, we obtain

$$\omega + 1.388,$$
$$\lambda + 1.591,$$
$$c + 0.549,$$

so that

$$F(x\,1.388 + 0.549) = 539.62\,\psi(x) + 0.6\,\Delta^4\psi.$$

The comparison between theory and observation is seen from diagram 15 The second term in the above expression has here been neglected.

Eighth Example. *Distribution of glands of swine, given in Example I, treated as belonging to type B of frequency curves.*

As has been already remarked, the frequency curve in this case may alternatively be regarded as belonging to type A or to type B. I have treated it before as an A-curve, and will now consider the same numbers belonging to a curve of type B.

Using the 4th method above, we obtain, according to the formulæ (25) and (29):

$$b - c = 2.082\ \sigma,$$
$$\lambda = 4.326,$$
$$\omega = 0.480\ \sigma,$$
$$\gamma_3 = -0.042,$$
$$\gamma_4 = -0.16.$$

Diagram 16 shows the comparison with the observations [1]). As might be expected, the agreement is somewhat closer at the discontinuous end than in example I, but, generally speaking, either curve may be used to represent the observations. Theoretically the curve B may be preferred.

Other examples of frequency curves belonging to type B may be gathered from different domains of statistics. I will confine myself, however, to the above given examples till two *desiderata* of the theory of these curves have been filled up. In the first place it is necessary to calculate a table of the function $\psi_\lambda(x)$, giving the values of this function for fractional values of the argument. In the second place it is necessary to calculate the error of the formula

$$\omega\, \Sigma\, (x\omega + c)^s F(x\omega + c) = \Sigma\, x^s F(x),$$

on which the computation of the parameters of the curve depends.

VI. Dissection of a frequency curve into components.

This problem has been first treated by Pearson. I have made during my lectures on the theory of probability this year some researches into this subject, and I will give here some extracts of the results obtained, reserving a fuller report till another opportunity.

Let us suppose that a given frequency curve is the resultant of two frequency curves belonging to the type A, with the corresponding values of β_3 and β_4 equal to zero. We hence have

(34) $$F(x) = c_1 \varphi_1 + c_2 \varphi_2,$$

[1]) In constructing the curve, the coefficients γ_3 and γ_4 have been neglected.

where c_1 and c_2 are certain constants and φ_1 and φ_2 are two normal curves, each with its special value of the coordinates of the mean (b_1 and b_2) and of the standard deviations σ_1 and σ_2.

Designating now with

$$\varphi = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-b)^2}{2\sigma^2}}$$

another normal-curve, we have, according to the general theory,

$$c_1\varphi_1 + c_2\varphi_2 = A_0\varphi + A_3\varphi^{\text{III}} + A_4\varphi^{\text{IV}} + \ldots, \tag{35}$$

b and σ being determined in such a manner, that A_1 and A_2 shall vanish.

The formula (26) in the »Meddelanden» N:o 27 gives us the following general expression of the coefficients A_r

$$A_r = \frac{\sigma^{2r}}{\underline{|r}} \int_{-\infty}^{+\infty} [c_1\varphi_1 + c_2\varphi_2] R_r(x)\, dx,$$

where $R_r(x)$ is given through formula (28*) in the same memoir.

Multiplying (35) successively by R_0, R_1, R_2, R_3, ... and integrating, we now obtain the following equations for determining the unknown quantities c_1, b_1, σ_1; c_2, b_2, σ_2. For the sake of convenience we have introduced the denominations

$$\tag{36} \begin{cases} b_1x_1 = b_1^2 + \sigma_1^2 - \sigma^2, & \\ b_2x_2 = b_2^2 + \sigma_2^2 - \sigma^2, & \\ y_1 = b_1, & z_1 = c_1, \\ y_2 = b_2, & z_2 = c_2, \\ & A_0\zeta_s = \underline{|s} A_s. \end{cases}$$

The equations now take the form

$$\tag{37} \begin{aligned} z_1 + {} & z_2 = 1, \\ y_1z_1 + {} & y_2z_2 = 0, \\ x_1y_1z_1 + {} & x_2y_2z_2 = 0, \\ y_1^2z_1(3x_1 - 2y_1) + {} & y_2^2z_2(3x_2 - 2y_2) = -\zeta_3, \\ y_1^2z_1(3x_1^2 - 2y_1^2) + {} & y_2^2z_2(3x_2^2 - 2y_2^2) = \zeta_4, \\ y_1^3z_1(15x_1^2 - 20x_1y_1 + 6y_1^2) + {} & y_2^3z_2(15x_2^2 - 20x_2y_2 + 6y_2^2) = -\zeta_5. \end{aligned}$$

From which equations the six unknown quantities x_1, y_1, z_1, x_2, y_2, z_2 are to be calculated. It is to be observed that ζ_3, ζ_4 and ζ_5 are known functions of the moments of the given frequency curve.

We have indeed

$$\zeta_s = \underline{|s}\, \sigma^s \beta_s, \tag{38}$$

where β_s (for $s = 3, 4, 5, \ldots$) are the characteristics of the frequency curve (Compare (5*)).

The solution of the above equations is dependent on a certain *nonic*, given by PEARSON.

We commence with the elimination of the quantities x_2, z_1 and z_2 by means of the relations

$$(39) \qquad \begin{aligned} x_2 &= x_1, \\ (y_1 - y_2) z_1 &= -y_2, \\ (y_1 - y_2) z_2 &= y_1. \end{aligned}$$

We then obtain the equations

$$\begin{aligned} y_1 y_2 [3x_1 - 2(y_1 + y_2)] &= \zeta_3, \\ y_1 y_2 [3x_1^2 - 2(y_1^2 + y_1 y_2 + y_2^2)] &= -\zeta_4, \\ y_1 y_2 [15 x_1^2 (y_1 + y_2) - 20 x_1 (y_1^2 + y_1 y_2 + y_2^2) + 6(y_1^3 + y_1^2 y_2 + y_1 y_2^2 + y_2^3)] &= \zeta_5. \end{aligned}$$

Putting

$$(40) \qquad \begin{aligned} u &= y_1 y_2, \\ w &= y_1 y_2 (y_1 + y_2), \end{aligned}$$

we obtain the fundamental equations

$$(41) \qquad \begin{cases} w - \zeta_3 = \dfrac{6\zeta_3 u^3 - 3\zeta_5 u^2 - 9\zeta_3 \zeta_4 u - 6\zeta_3^3}{2u^3 + 3\zeta_4 u + 4\zeta_3^2}, \\ 2(w - \zeta_3)^2 = 6u^3 + 3\zeta_4 u + 3\zeta_3^2. \end{cases}$$

Eliminating w between these equations we obtain the nonic of PEARSON:

$$(42) \qquad \begin{aligned} 0 = \; & 24u^9 + 84\zeta_4 u^7 + 36\zeta_3^2 u^6 + 72\zeta_3 \zeta_5 u^5 + 90\zeta_4^2 u^5 \\ & - 18\zeta_5^2 u^4 + 444\zeta_3^2 \zeta_4 u^4 + (288\zeta_3^4 - 108\zeta_3 \zeta_4 \zeta_5 + 27\zeta_4^3) u^3 \\ & - (63\zeta_3^2 \zeta_4^2 + 72\zeta_3^3 \zeta_5) u^2 - 96\zeta_3^4 \zeta_4 u - 24\zeta_3^6. \end{aligned}$$

When a root of this equation is found, we may calculate the corresponding value of w from either of the equations (41). The values of y_1 and y_2 are then equal to the roots of the equation

$$(43) \qquad y^2 - \frac{w}{u} y + u = 0.$$

The value of $x_1 = x_2$ is found from the equation

$$(44) \qquad 3ux_1 = 2w + \zeta_3.$$

Finally we get the values of z_1 and z_2 from (39). These equations are all linear with exception of (43). For obtaining real solutions from this equation it is necessary that the inequality

$$w^2 - 4u^3 > 0$$

is fulfilled. It may also be observed that for the reality of a solution it is necessary that the resulting values of σ_1^2 and σ_2^2 — obtained through the first two equations (36) — should be positive.

It is here supposed that we have solved the nonic (42). The solution of an equation of the ninth degree, where almost all powers, to the ninth, of the unknown quantity are existing, is, however, a very laborious operation. Mr Pearson has indeed possessed the energy to perform this heroic task in some instances in his first memoir on these topics from the year 1894. But I fear that he will have few successors, if the dissection of the frequency curve into two components is not very urgent.

A somewhat less tedious work may lead to the knowledge of the roots, if we start from the two equations (41).

Writing

$$(45)\qquad \begin{cases} U_1 = 6\zeta_3 u^3 - 3\zeta_5 u^2 - 9\zeta_3\zeta_4 u - 6\zeta_3^3, \\ U_2 = 2u^3 + 3\zeta_4 u + 4\zeta_3^2, \\ 2U_3 = 6u^3 + 3\zeta_4 u + 3\zeta_3^2, \end{cases}$$

we have

$$(46)\qquad \begin{cases} w - \zeta_3 = \dfrac{U_1}{U_2}, \\ (w - \zeta_3)^2 = U_3, \end{cases}$$

and here U_1, U_2 and U_3 are polynoms in u of the third degree. If the roots of the equations $U_1 = U_2 = U_3 = 0$ be known, the roots of the nonic may be easily discussed without solving the equation (42).

With this aim we construct the two curves defined by (46). We call them I and II. If

$$\begin{aligned} U_1 &= 6\zeta_3 (u - a_1)(u - a_2)(u - a_3), \\ U_2 &= 2 \quad (u - b_1)(u - b_2)(u - b_3), \\ U_3 &= 3 \quad (u - c_1)(u - c_2)(u - c_3), \end{aligned}$$

we find that I has infinite branches for $u = b_1$, $u = b_2$ and $u = b_3$. The curve II has generally a parabola-like appearance. Supposing c_1 and c_2 to be imaginary we have for instance the following form of the curves I and II — a_1, a_2, a_3 and b_1, b_2, b_3 being supposed to be all real.

We find from inspection that we must possess in this case 5 real roots of the nonic, the approximate values of which are directly found from the figure. For a more detailed knowledge of the roots we may calculate the curves more accurately in the neighbourhood of these approximate values.

I have applied this method to some instances and have found the determination of the values of the roots in this manner tolerably easy.

There is, however, enough labour left to discourage an inquirer from operating an mathematical dissection of a given frequency curve. In some instances the operation may be performed in an easier manner.

1:o *Suppose the values of* b_1 *and* b_2 *to be given.* The dissection of the frequency curve is then very easy. Using the same denominations as before ($b_1 = y_1$, $b_2 = y_2$ a. s. o.) we get z_1 and z_2 from the relations

$$\begin{aligned} (y_1 - y_2) z_1 &= -y_2, \\ (y_1 - y_2) z_2 &= \quad y_1 \end{aligned}$$

and, as $x_1 = x_2 = x$, we now only want an equation for x, which is

$$y_1 y_2 [3x - 2(y_1 + y_2)] = \zeta_3,$$

and the problem is solved.

This method is applicable, whenever the collective object consists of a mixture of two races (types), the mean value of the character in question being known for each of these types.

2:o *Suppose the given frequency curve to be symmetrical.* This case has been treated by PEARSON (1894). It is found that the two components are then either symmetrically situated to the mean and possess the same number of individuals, or that the two components have the same mean, coinciding with that of the frequency curve. In either case the solution is found through elementary operations.

3:o *Suppose the two components to possess equal standard deviations.*

Using the same abbreviations as before and putting

$$t = \sigma_1^2 - \sigma^2$$

we now have the equations

$$\begin{aligned} z_1 + z_2 &= 1, \\ b_1 z_1 + b_2 z_2 &= 0, \\ b_1^2 z_1 + b_2^2 z_2 &= -t, \\ b_1^3 z_1 + b_2^3 z_2 &= -\zeta_3, \\ b_1^4 z_1 + b_2^4 z_2 &= 3t^2 + \zeta_4. \end{aligned} \tag{47}$$

from which equations we may eliminate z_1, z_2, b_1 and b_2. The resulting equation for t is then

$$2t^3 + \zeta_4 t + \zeta_3^2 = 0. \tag{48}$$

When this equation is solved, we find b_1 and b_2 to be the roots of the quadratic

$$y^2 - \frac{\zeta_3}{t} y + t = 0. \tag{49}$$

Finally the values of z_1 and z_2 are found from the two first equations (47).

The supposition here made — that $\sigma_1 = \sigma_2$ — is of a more general character than those made in 1:o and 2:o. Especially in biology it is a fairly probable supposition that two types found together in the nature often possess *nearly* equal standard deviations. We may then use this method to separate the two components. We find for instance that the 19 pure lines of *Phaseolus vulgaris* cultivated by JOHANNSEN (compare table V) possess standard deviations that are surely not identical, but yet are of the same order. As an instance I have applied this method to the same curve, to which PEARSON first has applied his general method, namely the distribution of the frequency in the breadth of the head of 1000 Neapolitan crabs, measured by WELDON.

The equation (48) gave here, using the values of the moments obtained by PEARSON,

$$t = -11.32,$$

and hence is derived, taking the origin at the mean ($= +16.80$),

$$\begin{aligned} \sigma &= 3.38, \\ b_1 &= -6.50, \\ b_2 &= +1.74, \\ c_1 &= 212, \\ c_2 &= 788. \end{aligned}$$

The form of the components and of the resultant curve is shown from fig. 18, where I have used the same scale as PEARSON for facilitating the comparison with his curves. The value of σ lies between the values, found by PEARSON for the two components. Though his values are rather unequal, we find that the agreement in fig. 18 with the observed frequency curve is satisfactory.

I have applied this method also to artificial mixtures of different pure lines of the table V, and obtained acceptable results that at least may be used as a first approximation to a more accurate solution.

It is to be observed that the equation (48) coincides with the equation $U_3 = 0$, which is required for the general solution. Hence it is no loss of time to *begin* with this approximate method, which may be considered as an *abridged method for dissecting frequency curves.* It must be remarked that the problem of dissecting frequency curves into components is to a certain degree undetermined, there being a possibility of an infinity of solutions. Under such circumstances it is often not judicious to use too rigorous mathematical methods. Which may be understood in just the same manner as it is not judicious to use too many decimals in numerical calculations. It causes a temptation to overestimate the exactness of the result.

Naturally this »abridged method» is only applicable when there are *a priori reasons* for the assumption that the two components have nearly equal standard deviations. There are many problems, where no such reasons exist. If we consider for instance the frequency curve of the errors in astronomical transit observations, we may divide the perturbative sources of error into two different groups. On the one side we have the errors caused by psychological changes in the observer, on the other accidental changes in the instrument and in the environs. It is reasonable that the frequency curve may be considered as the resultant of two (normal) curves, representing respectively the subjective and the objective errors of observations. But there is no reason for the assumption that these two sources of errors should have equal or nearly equal standard deviations. In such a case there would be no meaning in the application of the abridged method.

I have endeavoured to obtain, with the help of ENGSTRÖM, materials for discussing the astronomical problem just now mentioned, which will no doubt furnish an excellent instance relating to the importance of the problem to dissect a frequency curve into unknown components. Up to this moment, however, I have not succeeded in getting a frequency curve with a sufficient number of individual observations.

I have extended the method here named the *abridged* one to the problem concerning the dissection of frequency curves into *three* components. The solution is then dependent on a certain *septic.*

It may occur also that there is reason to consider a given frequency curve as the *resultant of two curves of type B.* Such is for instance the case with many *multimodal* curves obtained in botany. The ray flowers of *Chrysanthemum segetum* belong to this class of curves, as may be found from some statistics gathered by HUGO DE VRIES and LUDWIG [1]). During this summer I have counted in a field (where peas were cultivated) the ray flowers of 1015 individuals of this flower. The result is shown from the following table.

Ninth Example. *Distribution of frequency of ray flowers of 1015 specimens of Chrysanthemum segetum.*

Number of ray flowers	8	9	10	11	12	13	14	15	16	17
Class	—5	—4	—3	—2	—1	0	+1	+2	+3	+4
Frequency	2	2	3	5	16	265	189	108	77	77

Number of ray flowers	18	19	20	21	22	23	24
Class	+5	+6	+7	+8	+9	+10	+11
Frequency	57	66	56	88	2	1	1

It is very probable that we here have to do with a composite frequency curve, consisting of two curves of type *B*, the one having its summit at 13 rays the other at 21. Fig. 19 shows how these components *could* be constituted. A biological research here can give a definite answer [2]).

For solving such a problem we can proceed in the following manner, that may be considered only as a preliminary to a definite solution.

Calling the x coordinates of the *summits* of the components c_1 and c_2, and designating with k_1 and k_2 two unknown constants, we may write the frequency curve in the form

$$F(x) = k_1 \psi_1 (x - c_1) + k_2 \psi_2 (c_2 - x), \tag{50}$$

where ψ_1 and ψ_2 with the characteristics λ_1 and λ_2 respectively designate two curves of type B. More generally we may consider the scales ω_1 and ω_2 different (and differing from unity) for the two curves. Limiting ourselves to the form (50), we may consider c_1 and c_2 as known (coinciding with the x coordinate for 13 and 21 ray flowers in fig. 19), and hence have four constants λ_1, λ_2, k_1 and k_2 to determine from the frequency curve.

[1]) Compare the bibliography in DAVENPORT's »Statistical Methods».

[2]) If the collection of flowers in question should be composed in the manner indicated by the figure, it follows that the offspring of plants with 23 and 24 ray flowers would generally belong to the 13-type, whereas plants with 11, 10, 9 and 8 ray flowers should give rise to an offspring belonging to the 21-type.

Choosing the *mean* of the given frequency curve as the origin of the coordinates, we obtain through multiplication by 1, x, x^2 and x^3 and adding the equations of condition

$$
\begin{aligned}
\mu_0 &= k_1 + k_2, \\
0 &= k_1 \Sigma x\psi_1(x - c_1) + k_2 \Sigma x\psi_2(c_2 - x), \\
\mu_2 &= k_1 \Sigma x^2 \psi_1(x - c_1) + k_2 \Sigma x^2 \psi_2(c_2 - x), \\
\mu_3 &= k_1 \Sigma x^3 \psi_1(x - c_1) + k_2 \Sigma x^3 \psi_3(c_2 - x).
\end{aligned} \tag{51}
$$

Now we have

$$
\begin{aligned}
\Sigma x^s \psi_1(x - c_1) &= \Sigma (c_1 + y)^s \psi_1(y) \\
&= c_1^s \Sigma \psi_1 + \binom{s}{1} c_1^{s-1} \Sigma y\psi_1 + \binom{s}{2} c_1^{s-2} \Sigma y^2 \psi_1 + \ldots,
\end{aligned}
$$

and in like manner

$$\Sigma x^s \psi_2(c_2 - x) = c_2^s \Sigma \psi_2 - \binom{s}{1} c_2^{s-1} \Sigma y\psi_2 + \binom{s}{2} c_2^{s-2} \Sigma y^2 \psi_2 + \ldots$$

But

$$
\begin{aligned}
\Sigma \psi &= 1, \\
\Sigma y\psi &= \lambda, \\
\Sigma y^2 \psi &= \lambda^2 + \lambda, \\
\Sigma y^3 \psi &= \lambda^3 + 3\lambda^2 + \lambda,
\end{aligned}
$$

and hence we have

$$
\begin{aligned}
\Sigma x\psi_1(x - c_1) &= c_1 + \lambda_1, \\
\Sigma x^2 \psi_1(x - c_1) &= c_1^2 + 2c_1\lambda_1 + \lambda_1^2 + \lambda_1, \\
\Sigma x^3 \psi_1(x - c_1) &= c_1^3 + 3c_1^2\lambda_1 + 3c_1(\lambda_1^2 + \lambda_1) + \lambda_1^3 + 3\lambda_1^2 + \lambda_1,
\end{aligned}
$$

and corresponding expressions for $\Sigma x^s \psi_2(c_2 - x)$.

The equations (51) thus take the form

$$
\begin{aligned}
\mu_0 &= k_1 + k_2, \\
0 &= k_1[c_1 + \lambda_1] + k_2[c_2 - \lambda_2], \\
\mu_2 &= k_1[c_1^2 + 2c_1\lambda_1 + \lambda_1^2 + \lambda_1] + k_2[c_2^2 - 2c_2\lambda_2 + \lambda_2^2 + \lambda_2], \\
\mu_3 &= k_1[c_1^3 + 3c_1^2\lambda_1 + 3c_1(\lambda_1^2 + \lambda_1) + \lambda_1^3 + 3\lambda_1^2 + \lambda_1] \\
&\quad + k_2[c_2^3 - 3c_2^2\lambda_2 + 3c_2(\lambda_2^2 + \lambda_2) - \lambda_2^3 - 3\lambda_2^2 - \lambda_2].
\end{aligned}
$$

From the first two equations we get

$$
\begin{aligned}
(c_2 - c_1 - \lambda_1 - \lambda_2)k_1 &= + \mu_0(c_2 - \lambda_2), \\
(c_2 - c_1 - \lambda_1 - \lambda_2)k_2 &= - \mu_0(c_1 + \lambda_1),
\end{aligned} \tag{52}
$$

which expressions substituted in the latter two equations give us the relations

$$
\begin{aligned}
\nu_2(c_2 - c_1 - \lambda_1 - \lambda_2) &= (c_2 - \lambda_2)[c_1^2 + 2c_1\lambda_1 + \lambda_1^2 + \lambda_1] - (c_1 + \lambda_1)[c_2^2 - 2c_2\lambda_2 + \lambda_2^2 + \lambda_2], \\
\nu_3(c_2 - c_1 - \lambda_1 - \lambda_2) &= (c_2 - \lambda_2)[c_1^3 + 3c_1^2\lambda_1 + 3c_1(\lambda_1^2 + \lambda_1) + \lambda_1^3 + 3\lambda_1^2 + \lambda_1] \\
&\quad - (c_1 + \lambda_1)[c_2^3 - 3c_2^2\lambda_2 + 3c_2(\lambda_2^2 + \lambda_2) - \lambda_2^3 - 3\lambda_2^2 - \lambda_2].
\end{aligned}
$$

I do not know, if these equations can be algebraically solved (h. e. reduced to the 4^{th} degree). They may be numerically discussed, though somewhat laboriously. It seems, however, advisable to take another course.

In many cases the maximum ordinate of the two components may be considered as known with a good approximation. Calling these ordinates y_1 and y_2 we thus get the relations

$$y_1 = k_1 e^{-\lambda_1}, \quad y_2 = k_2 e^{-\lambda_2}, \tag{53}$$

by means of which k_1 and k_2 may be eliminated from the equations of condition. It is too possible in this manner to attack the problem somewhat more generally. We may write

$$F(x) = f_1(x) + f_2(x),$$

where

$$f_1(x) = B_0' \psi_1(x) + B_2' \Delta^2 \psi_1,$$
$$f_2(x) = B_0'' \psi_1(x) + B_2'' \Delta^2 \psi_2,$$

or we can make use of another scale than unity, one for each function (say ω_1 and ω_2).

Should it be allowable to put $B_2' = B_2'' = 0$ (or $\omega_1 = \omega_2 = 1$), we get the relations

$$\begin{aligned} \mu_0 &= y_1 e^{\lambda_1} + y_2 e^{\lambda_2}, \\ 0 &= y_1 e^{\lambda_1}(c_1 + \lambda_1) + y_2 e^{\lambda_2}(c_2 - \lambda_2). \end{aligned} \tag{54}$$

These equations indeed are of transcendental nature, but may easily be discussed with the help of graphical methods.

To the tables and diagrams.

Tab. A and Tab. B contain the values, to four decimals, of the probability integral and of the probability function in the form used in this memoir. These tables are extracted from the »New tables of the probability integral» by *W. F. Sheppard* in »Biometrika» Vol. II (1903).

Tab. C and Tab. D give the values of the functions φ_3 and φ_4, used in the formula for frequency curves of type *A*. The expression of the frequency is

$$\sigma F(x) = \mu_0 [\varphi_0(x) + \beta_3\varphi_3 + \beta_4\varphi_4 + \ldots].$$

Tab. E gives the value of the function $\psi_\lambda(x)$, used in the formulæ for frequency curves o type *B*, for *integer* values of x. For such values we have

$$\psi_\lambda(x) = \frac{e^{-\lambda}\lambda^x}{\underline{|x}},$$

which function is tabulated in the memoir of *Bortkewitsch* »Das Gesetz der kleinen Zahlen», from which this table is extracted.

Fig. 1. Normal curve, $\beta_3 = \beta_4 = 0$.
» 2. Frequency curve with positive skewness, $\beta_3 = +0.1$, $\beta_4 = 0$.
» 3. Frequency curve with positive skewness, $\beta_3 = +0.2$, $\beta_4 = 0$.
» 4. Frequency curve with positive excess, $\beta_3 = 0$, $\beta_4 = +0.1$.
» 5. Frequency curve with negative excess, $\beta_3 = 0$, $\beta_4 = -0.1$.
» 6. Frequency of glands in the leg of female swine (Davenport).
» 7. Frequency of stigmatic bands of papaver (Charlier).
» 8. Line A of brown beans (Johannsen).
» 9. Line G of brown beans (Johannsen).
» 10. Line 0 of brown beans (Johannsen).
» 11. Cephalic Index of Swedish recruits (Retzius and Fürst).
» 12. Typhoid Fever in Lund 1905 (Ryberg).
» 13. Frequency curves of type B.
» 14. Frequency of Petals of Ranunculus bulbosus (de Vries).
» 15. Failing percentage of barley (Johannsen).
» 16. Frequency of glands of swine treated as a B-curve.
» 17. Dissection of frequency curves.
» 18. Breadth of »forehead» of Naples crabs (Weldon).
» 19. Distribution of frequency of ray flowers of 1015 samples of Chrysanthemum segetum (Charlier).

The *observed* values are in all diagrams indicated by small circles.

TAB. A. *Table of the Probability Integral* $\frac{2}{\sqrt{2\pi}}\int_0^{\frac{x}{\sigma}} e^{-\frac{t^2}{2}}\,dt = \frac{2}{\sigma\sqrt{2\pi}}\int_0^{x} e^{-\frac{u^2}{2\sigma^2}}\,du.$

$\frac{x}{\sigma}$	0	1	2	3	4	5	6	7	8	9
0.0	0000	0080	0159	0239	0319	0399	0478	0558	0637	0717
0.1	0796	0876	0955	1034	1113	1192	1271	1350	1428	1507
0.2	1585	1663	1741	1819	1897	1974	2051	2128	2205	2282
0.3	2358	2434	2510	2586	2661	2737	2811	2886	2960	3035
0.4	3108	3181	3255	3328	3400	3473	3545	3616	3688	3759
0.5	3829	3899	3969	4039	4108	4177	4245	4313	4381	4448
0.6	4515	4581	4647	4713	4778	4843	4907	4971	5035	5098
0.7	5161	5223	5285	5346	5407	5467	5527	5587	5646	5705
0.8	5763	5820	5878	5935	5991	6047	6102	6157	6211	6265
0.9	6319	6372	6424	6476	6528	6579	6629	6679	6729	6778
1.0	6827	6875	6923	6970	7017	7063	7109	7154	7198	7243
1.1	7287	7330	7373	7415	7457	7499	7540	7580	7620	7660
1 2	7699	7737	7775	7813	7850	7887	7823	7959	7995	8029
1.3	8064	8098	8132	8165	8198	8230	8262	8293	8324	8355
1.4	8385	8415	8444	8473	8501	8529	8557	8584	8611	8638
1.5	8664	8689	8715	8740	8764	8788	8812	8836	8859	8882
1.6	8904	8926	8948	8969	8990	9011	9031	9051	9070	9089
1.7	9109	9127	9146	9164	9181	9199	9216	9233	9249	9265
1.8	9281	9297	9312	9327	9342	9357	9371	9385	9399	9412
1.9	9426	9439	9451	9464	9476	9488	9500	9512	9523	9534
2.0	9545	9556	9566	9576	9586	9596	9606	9615	9625	9634
2.1	9643	9651	9660	9668	9676	9684	9692	9700	9707	9715
2.2	9722	9729	9736	9742	9749	9755	9762	9768	9774	9780
2.3	9786	9791	9797	9802	9807	9812	9817	9822	9827	9832
2.4	9836	9840	9845	9849	9853	9857	9861	9865	9869	9872
2.5	9876	9879	9883	9886	9889	9892	9895	9898	9901	9904
2.6	9907	9909	9912	9915	9917	9919	9922	9924	9926	9928
2.7	9931	9933	9935	9937	9939	9940	9942	9944	9946	9947
2.8	9949	9950	9952	9953	9955	9956	9958	9959	9960	9961
2.9	9963	9964	9965	9966	9967	9968	9969	9970	9971	9972
3.	9973	9981	9986	9990	9993	9995	9997	9998	9998	9999
4.	^{9}9994	9996	9997	9998	9999	^{99}9993	9996	9997	9998	9999
5.	^{9999}9943	9966	9980	9988	9993	9996	9998	9999	^{99999}9993	9996
6.	^{99999}9998									
$\frac{x}{\sigma}$	0	1	2	3	4	5	6	7	8	9

TAB. B. *Table of the function* $\varphi_0 = \sigma\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-b)^2}{2\sigma^2}}$.

$\frac{x-b}{\sigma}$	0	1	2	3	4	5	6	7	8	9	Δ
0.0	.3989	3989	3989	3988	3986	3984	3982	3980	3977	3973	— 3
0.1	.3970	3965	3961	3956	3951	3945	3939	3932	3925	3918	— 8
0.2	.3910	3902	3894	3885	3876	3867	3857	3847	3836	3825	— 11
0.3	.3814	3802	3790	3778	3765	3752	3739	3725	3712	3697	— 14
0.4	.3683	3668	3653	3637	3621	3605	3589	3572	3555	3538	— 17
0.5	.3521	3503	3485	3467	3448	3429	3410	3391	3372	3352	— 20
0.6	.3332	3312	3292	3271	3251	3230	3209	3187	3166	3144	— 21
0.7	.3123	3101	3079	3056	3034	3011	2989	2966	2943	2920	— 23
0.8	.2897	2874	2850	2827	2803	2780	2756	2732	2709	2685	— 24
0.9	.2661	2637	2613	2589	2565	2541	2516	2492	2468	2444	— 24
1.0	.2420	2396	2371	2347	2323	2299	2275	2251	2227	2203	— 24
1.1	.2179	2155	2131	2107	2083	2059	2036	2012	1989	1965	— 23
1.2	.1942	1919	1895	1872	1849	1826	1804	1781	1758	1736	— 22
1.3	.1714	1691	1669	1647	1626	1604	1582	1561	1539	1518	— 21
1.4	.1497	1476	1456	1435	1415	1394	1374	1354	1334	1315	— 20
1.5	.1295	1276	1257	1238	1219	1200	1182	1163	1145	1127	— 18
1.6	.1109	1092	1074	1057	1040	1023	1006	0989	0973	0957	— 17
1.7	.0940	0925	0909	0893	0878	0863	0848	0833	0818	0804	— 14
1.8	.0790	0775	0761	0748	0734	0721	0707	0694	0681	0669	— 13
1.9	.0656	0644	0632	0620	0608	0596	0584	0573	0562	0551	— 11
2.0	.0540	0529	0519	0508	0498	0488	0478	0468	0459	0449	— 9
2.1	.0440	0431	0422	0413	0404	0396	0387	0379	0371	0363	— 8
2.2	.0355	0347	0339	0332	0325	0317	0310	0303	0297	0290	— 7
2.3	.0283	0277	0270	0264	0258	0252	0246	0241	0235	0229	— 5
2.4	.0224	0219	0213	0208	0203	0198	0194	0189	0184	0180	— 5
2.5	.0175	0171	0167	0163	0158	0154	0151	0147	0143	0139	— 3
2.6	.0136	0132	0129	0126	0122	0119	0116	0113	0110	0107	— 3
2.7	.0104	0101	0099	0096	0093	0091	0088	0086	0084	0081	— 2
2.8	.0079	0077	0075	0073	0071	0069	0067	0065	0063	0061	— 1
2.9	.0060	0058	0056	0055	0053	0051	0050	0048	0047	0046	— 2
3.	.0044	0033	0024	0017	0012	0009	0006	0004	0003	0002	— 1
4.	.0001	0001	0001	0000	0000	0000	0000	0000	0000	0000	
$\frac{x-b}{\sigma}$	0	1	2	3	4	5	6	7	8	9	Δ

TAB. C. *Table of the function* $\varphi_3 = \sigma^4 \varphi'''(x)$.

N.B.! Permutation of sign at the argument 1.73!

$\frac{x-b}{\sigma}$	0	1	2	3	4	5	6	7	8	9	Δ
0.0	+ .0000	0120	0239	0359	0478	0597	0716	0834	0952	1070	+117
0.1	+ .1187	1303	1419	1534	1648	1762	1874	1986	2097	2206	+109
0.2	+ .2315	2422	2529	2634	2737	2840	2941	3040	3188	3235	+ 95
0.3	+ .3330	3423	3514	3604	3693	3779	3864	3947	4028	4106	+ 78
0.4	+ .4184	4259	4332	4403	4472	4539	4603	4666	4726	4785	+ 56
0.5	+ .4841	4895	4946	4996	5043	5088	5131	5171	5209	5245	+ 33
0.6	+ .5278	5309	5338	5365	5389	5411	5431	5448	5463	5476	+ 10
0.7	+ .5486	5495	5501	5504	5506	5505	5502	5497	5490	5481	— 12
0.8	+ .5469	5456	5440	5423	5403	5381	5358	5332	5305	5276	— 31
0.9	+ .5245	5212	5177	5140	5102	5062	5021	4978	4933	4887	— 48
1.0	+ .4839	4790	4740	4688	4635	4580	4524	4467	4409	4350	— 60
1.1	+ .4290	4228	4166	4102	4038	3973	3907	3840	3772	3704	— 69
1.2	+ .3635	3566	3495	3425	3354	3282	3210	3138	3065	2992	— 74
1.3	+ .2918	2845	2771	2697	2623	2549	2476	2402	2328	2254	— 74
1.4	+ .2180	2106	2033	1960	1887	1815	1742	1670	1599	1528	— 71
1.5	+ .1457	1387	1317	1248	1179	1111	1044	0977	0911	0846	— 65
1.6	+ .0781	0717	0654	0591	0529	0468	0408	0349	0290	0233	— 57
1.7	+ .0176	0120	0065	0011	—0042	0094	0146	0196	0245	0294	— 47
1.8	— .0341	0387	0433	0477	0521	0563	0605	0645	0685	0723	— 37
1.9	— .0760	0797	0832	0867	0900	0933	0964	0994	1024	1052	— 28
2.0	— .1080	1106	1132	1156	1180	1203	1225	1245	1265	1284	— 18
2.1	— .1302	1320	1336	1351	1366	1380	1393	1405	1416	1426	— 10
2.2	— .1436	1445	1453	1460	1467	1473	1478	1483	1486	1490	— 2
2.3	— .1492	1494	1495	1496	1496	1495	1494	1492	1490	1487	+ 4
2.4	— .1483	1480	1475	1470	1465	1459	1453	1446	1439	1432	+ 8
2.5	— .1424	1416	1407	1398	1389	1380	1370	1360	1349	1339	+ 11
2.6	— .1328	1317	1305	1294	1282	1270	1258	1245	1233	1220	+ 13
2.7	— .1207	1194	1181	1168	1154	1141	1127	1114	1100	1086	+ 13
2.8	— .1073	1059	1045	1031	1017	1003	0989	0976	0962	0948	+ 14
2.9	— .0934	0920	0906	0892	0879	0865	0852	0838	0824	0811	+ 13
3.	— .0798	0669	0552	0449	0359	0283	0219	0168	0127	0095	+ 25
4.	— .0070	0051	0036	0026	0018	0012	0008	0006	0004	0002	
5.	— .0002	0001	00010	00007	00004	00003	00002	00001	00000	00000	
6.	.00000										
$\frac{x-b}{\sigma}$	0	1	2	3	4	5	6	7	8	9	Δ

TAB. D. *Table of the function* $\varphi_4 = \sigma^5 \varphi^{IV}(x)$.

N.B.! Permutations of sign at the arguments 0.74 and 2.33.

$\frac{x-b}{\sigma}$	0	1	2	3	4	5	6	7	8	9	Δ
0.0	+1.1968	1.1965	1.1956	1.1941	1.1920	1.1894	1.1861	1.1822	1.1778	1.1727	— 56
0.1	+1.1671	1.1609	1.1541	1.1468	1.1388	1.1304	1.1214	1.1118	1.1017	1.0911	—112
0.2	+1.0799	1.0682	1.0560	1.0434	1.0302	1.0165	1.0024	0.9878	0.9727	0.9572	—159
0.3	+ .9413	9250	9082	8910	8735	8555	8373	8186	7996	7803	—196
0.4	+ .7607	7408	7206	7001	6793	6583	6371	6156	5940	5721	—220
0.5	+ .5501	5279	5056	4831	4605	4378	4150	3921	3691	3461	—230
0.6	+ .3231	3000	2770	2539	2308	2078	1849	1619	1391	1164	—227
0.7	+ .0937	0712	0487	0265	0043	—0176	0394	0611	0825	1037	—210
0.8	— .1247	1454	1660	1862	2063	2260	2455	2645	2835	3021	—182
0.9	— .3203	3383	3559	3731	3901	4066	4228	4387	4541	4692	—147
1.0	— .4839	4983	5122	5257	5389	5516	5639	5758	5873	5984	—107
1.1	— .6091	6193	6292	6386	6476	6561	6642	6720	6792	6861	— 64
1.2	— .6925	6986	7042	7093	7141	7185	7224	7259	7291	7318	— 23
1.3	— .7341	7361	7376	7388	7395	7399	7400	7396	7389	7378	+ 14
1.4	— .7364	7347	7326	7301	7274	7243	7209	7172	7132	7088	+ 46
1.5	— .7042	6994	6942	6888	6831	6772	6710	6646	6580	6511	+ 71
1.6	— .6440	6368	6293	6216	6138	6057	5975	5892	5806	5720	+ 88
1.7	— .5632	5542	5452	5360	5267	5173	5078	4983	4886	4789	+ 97
1.8	— .4692	4593	4494	4395	4295	4195	4095	3995	3894	3793	+100
1.9	— .3693	3592	3492	3392	3292	3192	3092	2994	2895	2797	+ 97
2.0	— .2700	2603	2506	2411	2316	2222	2129	2036	1945	1854	+ 90
2.1	— .1764	1676	1588	1502	1416	1332	1249	1166	1086	1006	+ 79
2.2	— .0927	0850	0774	0700	0626	0554	0483	0414	0346	0279	+ 65
2.3	— .0214	0150	0088	0027	+0033	0092	0148	0204	0258	0311	+ 51
2.4	+ .0362	0412	0461	0508	0554	0598	0641	0683	0723	0762	+ 38
2.5	+ .0800	0836	0871	0905	0937	0968	0998	1027	1054	1080	+ 25
2.6	+ .1105	1129	1152	1173	1193	1213	1231	1248	1264	1279	+ 14
2.7	+ .1293	1306	1317	1328	1338	1347	1355	1363	1369	1375	+ 4
2.8	+ .1379	1383	1386	1389	1390	1391	1391	1391	1389	1388	— 3
2.9	+ .1385	1382	1378	1374	1369	1364	1358	1351	1345	1337	— 7
3.	+ .1330	1231	1107	0969	0829	0694	0570	0460	0364	0284	— 66
4.	+ .0218	0165	0123	0090	0065	0047	0033	0023	0016	0011	— 4
5.	+ .0007	0005	0003	0002	0001	0001	0000	0000	0000	0000	0
6.	.0000										
$\frac{x-b}{\sigma}$	0	1	2	3	4	5	6	7	8	9	Δ

TAB. E. *Table of the function* $\psi_\lambda(x)$

for integer values of x.

λ=	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
x=0	.9048	8187	7408	6703	6065	5488	4966	4493	4066	3679
1	.0905	1637	2223	2681	3033	3293	3476	3595	3659	3679
2	.0045	0164	0333	0536	0758	0988	1217	1438	1647	1839
3	.0002	0011	0033	0072	0126	0198	0284	0383	0494	0613
4		0001	0003	0007	0016	0030	0050	0077	0111	0153
5				0001	0002	0004	0007	0012	0020	0031
6							0001	0002	0003	0005
7										0001

λ=	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
x=0	.3329	3012	2725	2466	2231	2019	1827	1653	1496	1353
1	.3662	3614	3543	3452	3347	3230	3106	2975	2842	2707
2	.2014	2169	2303	2417	2510	2584	2640	2678	2700	2707
3	.0738	0867	0998	1128	1255	1378	1496	1607	1710	1804
4	.0203	0260	0324	0395	0471	0551	0636	0723	0812	0902
5	.0045	0062	0084	0111	0141	0176	0216	0260	0309	0361
6	.0008	0012	0018	0026	0035	0047	0061	0078	0098	0120
7	.0001	0002	0003	0005	0008	0011	0015	0020	0027	0034
8			0001	0001	0001	0002	0003	0005	0006	0009
9							0001	0001	0001	0002

λ=	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
x= 0	.1225	1108	1003	0907	0821	0743	0672	0608	0550	0498
1	.2572	2438	2306	2177	2052	1931	1815	1703	1596	1494
2	.2700	2681	2652	2613	2565	2510	2450	2384	2314	2240
3	.1890	1966	2033	2090	2138	2176	2205	2225	2237	2240
4	.0992	1082	1169	1254	1336	1414	1488	1557	1622	1680
5	.0417	0476	0538	0602	0668	0735	0804	0872	0941	1008
6	.0146	0175	0206	0241	0278	0319	0362	0407	0455	0504
7	.0044	0055	0068	0083	0099	0118	0140	0163	0188	0216
8	.0011	0015	0019	0025	0031	0038	0047	0057	0068	0081
9	.0003	0004	0005	0007	0009	0011	0014	0018	0022	0027
10	.0001	0001	0001	0001	0002	0003	0004	0005	0006	0007
11						0001	0001	0001	0002	0001
12										

λ=	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9	4.0
x= 0	.0451	0408	0369	0334	0302	0273	0247	0224	0202	0183
1	.1397	1304	1217	1135	1057	0984	0915	0850	0789	0733
2	.2165	2087	2008	1929	1850	1771	1692	1615	1539	1465
3	.2237	2226	2209	2186	2158	2125	2087	2046	2001	1954
4	.1733	1781	1822	1858	1888	1912	1931	1944	1951	1954
5	.1075	1140	1203	1264	1322	1377	1429	1477	1522	1563
6	.0555	0608	0662	0716	0771	0826	0881	0936	0989	1042
7	.0246	0278	0312	0348	0386	0425	0466	0508	0551	0595
8	.0095	0111	0129	0148	0169	0191	0215	0241	0269	0298
9	.0033	0040	0047	0056	0066	0076	0089	0102	0116	0132
10	.0010	0013	0016	0019	0023	0028	0033	0039	0045	0053
11	.0003	0004	0005	0006	0007	0009	0011	0013	0016	0019
12	.0001	0001	0001	0002	0002	0003	0003	0003	0005	0006
13					0001	0001	0001	0001	0002	0002
14										0001

λ=	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9	5.0
x= 0	.0166	0150	0136	0123	0111	0101	0091	0082	0074	0067
1	.0680	0630	0584	0540	0500	0462	0428	0395	0365	0337
2	.1393	1323	1254	1188	1125	1063	1005	0948	0894	0842
3	.1904	1852	1798	1743	1687	1631	1574	1517	1460	1404
4	.1951	1944	1933	1917	1898	1875	1849	1820	1789	1755
5	.1600	1633	1662	1687	1708	1725	1738	1747	1753	1755
6	.1093	1143	1191	1237	1281	1323	1362	1398	1432	1462
7	.0640	0686	0732	0778	0824	0869	0914	0959	1002	1044
8	.0328	0360	0393	0428	0463	0500	0537	0575	0614	0653
9	.0150	0168	0188	0209	0232	0256	0281	0307	0334	0363
10	.0061	0071	0081	0092	0104	0118	0132	0147	0164	0181
11	.0023	0027	0032	0037	0043	0049	0056	0064	0073	0082
12	.0008	0009	0011	0014	0016	0019	0022	0026	0030	0034
13	.0002	0003	0004	0005	0006	0007	0008	0009	0011	0013
14	.0001	0001	0001	0001	0002	0002	0003	0003	0004	0005
15					0001	0001	0001	0001	0001	0002

λ=	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9	6.0
x=0	.0061	0055	0050	0045	0041	0037	0033	0030	0027	0025
1	.0311	0287	0265	0244	0225	0207	0191	0176	0162	0149
2	.0793	0746	0701	0659	0618	0580	0544	0509	0477	0446
3	.1348	1293	1239	1185	1133	1082	1033	0985	0938	0892
4	.1719	1681	1641	1600	1558	1515	1472	1428	1383	1339
5	.1753	1748	1740	1728	1714	1697	1678	1656	1632	1606
6	.1490	1515	1537	1555	1571	1584	1594	1601	1605	1606
7	.1086	1125	1163	1200	1234	1267	1298	1326	1353	1377
8	.0692	0732	0771	0810	0849	0887	0925	0962	0998	1033

λ=	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9	6.0
$x=$ 9	.0392	0423	0454	0486	0519	0552	0586	0620	0654	0688
10	.0200	0220	0241	0262	0285	0309	0334	0359	0386	0413
11	.0093	0104	0116	0129	0143	0157	0173	0190	0207	0225
12	.0039	0045	0051	0058	0065	0073	0082	0092	0102	0113
13	.0015	0018	0021	0024	0028	0032	0036	0041	0046	0052
14	.0006	0007	0008	0009	0011	0013	0015	0017	0019	0022
15	.0002	0002	0003	0003	0004	0005	0006	0007	0007	0009
16	.0001	0001	0001	0001	0001	0002	0002	0002	0002	0003
17						0001	0001	0001	0001	0001

λ=	6.1	6.2	6.3	6.4	6.5	6.6	6.7	6.8	6.9	7.0
$x=$ 0	.0022	0020	0018	0017	0015	0014	0012	0011	0010	0009
1	.0137	0126	0116	0106	0098	0090	0082	0076	0070	0064
2	.0417	0390	0364	0340	0318	0296	0276	0258	0240	0223
3	.0849	0806	0765	0726	0688	0652	0617	0584	0552	0521
4	.1294	1249	1205	1162	1118	1076	1034	0992	0952	0912
5	.1579	1549	1519	1487	1453	1420	1385	1349	1314	1277
6	.1605	1601	1595	1586	1575	1562	1547	1529	1511	1490
7	.1399	1418	1435	1450	1462	1472	1480	1486	1489	1490
8	.1066	1099	1130	1160	1188	1215	1240	1263	1284	1304
9	.0723	0757	0791	0825	0858	0891	0923	0954	0985	1014
10	.0441	0469	0498	0528	0558	0588	0618	0649	0679	0710
11	.0245	0265	0286	0307	0330	0353	0377	0401	0426	0452
12	.0124	0137	0150	0164	0179	0194	0210	0227	0245	0264
13	.0058	0065	0073	0081	0089	0099	0108	0119	0130	0142
14	.0025	0029	0033	0037	0041	0046	0052	0058	0064	0071
15	.0010	0012	0014	0016	0018	0020	0023	0026	0029	0033
16	.0004	0005	0005	0006	0007	0008	0010	0011	0013	0014
17	.0001	0002	0002	0002	0003	0003	0004	0004	0005	0006
18		0001	0001	0001	0001	0001	0001	0002	0002	0002
19							0001	0001	0001	0001

λ=	7.1	7.2	7.3	7.4	7.5	7.6	7.7	7.8	7.9	8.0
$x=$ 0	.0008	0007	0007	0006	0006	0005	0005	0004	0004	0003
1	.0059	0054	0049	0045	0041	0038	0035	0032	0029	0027
2	.0208	0194	0180	0167	0156	0145	0134	0125	0116	0107
3	.0492	.0464	0438	0413	0389	0366	0345	0324	0305	0286
4	.0874	0836	0799	0764	0729	0696	0663	0632	0602	0573
5	.1241	1204	1167	1130	1094	1057	1021	0986	0951	0916
6	.1468	1445	1420	1394	1367	1340	1311	1282	1252	1221
7	.1489	1486	1481	1474	1465	1454	1442	1428	1413	1396
8	.1321	1337	1351	1363	1373	1382	1388	1392	1395	1396
9	.1042	1070	1096	1121	1144	1167	1187	1207	1225	1241
10	.0740	0770	0800	0829	0858	0887	0914	0941	0967	0993

λ=	7.1	7.2	7.3	7.4	7.5	7.6	7.7	7.8	7.9	8.0
x=11	.0478	0504	0531	0558	0585	0613	0640	0667	0695	0722
12	.0283	0303	0323	0344	0366	0388	0411	0434	0457	0481
13	.0154	0168	0181	0196	0211	0227	0243	0260	0278	0296
14	.0078	0086	0095	0104	0113	0123	0134	0145	0157	0169
15	.0037	0041	0046	0051	0057	0062	0069	0075	0083	0090
16	.0016	0019	0021	0024	0026	0030	0033	0037	0041	0045
17	.0007	0008	0009	0010	0012	0013	0015	0017	0019	0021
18	.0003	0003	0004	0004	0005	0006	0006	0007	0008	0009
19	.0001	0001	0001	0002	0002	0002	0003	0003	0003	0004
20			0001	0001	0001	0001	0001	0001	0001	0002
21									0001	0001

λ=	8.1	8.2	8.3	8.4	8.5	8.6	8.7	8.8	8.9	9.0
x= 0	.0003	0003	0002	0002	0002	0002	0002	0002	0002	0001
1	.0025	0023	0021	0019	0017	0016	0014	0013	0012	0011
2	.0100	0092	0086	0079	0074	0068	0063	0058	0054	0050
3	.0269	0252	0237	0222	0208	0195	0183	0171	0160	0150
4	.0544	0517	0491	0467	0443	0420	0398	0377	0357	0337
5	.0882	0849	0816	0784	0752	0722	0692	0663	0635	0607
6	.1191	1160	1128	1097	1066	1035	1003	0972	0941	0911
7	.1378	1358	1338	1317	1294	1271	1247	1222	1197	1171
8	.1395	1392	1388	1383	1375	1366	1356	1344	1332	1318
9	.1256	1269	1280	1291	1299	1306	1311	1315	1317	1318
10	.1017	1040	1063	1084	1104	1123	1140	1157	1172	1186
11	.0749	0776	0802	0828	0853	0878	0902	0926	0948	0970
12	.0506	0530	0555	0580	0604	0629	0654	0679	0703	0728
13	.0315	0334	0354	0374	0395	0416	0438	0459	0482	0504
14	.0182	0196	0210	0225	0240	0256	0272	0289	0306	0324
15	.0098	0107	0116	0126	0136	0147	0158	0169	0182	0194
16	.0050	0055	0060	0066	0072	0079	0086	0093	0101	0109
17	.0024	0026	0029	0033	0036	0040	0044	0048	0053	0058
18	.0011	0012	0014	0015	0017	0019	0021	0024	0026	0029
19	.0005	0005	0006	0007	0008	0009	0010	0011	0012	0014
20	.0002	0002	0002	0003	0003	0004	0004	0005	0005	0006
21	.0001	0001	0001	0001	0001	0002	0002	0002	0002	0003
22					0001	0001	0001	0001	0001	0001

λ=	9.1	9.2	9.3	9.4	9.5	9.6	9.7	9.8	9.9	10.0
x=0	.0001	0001	0001	0001	0001	0001	0001	0001	0001	
1	.0010	0009	0009	0008	0007	0007	0006	0005	0005	0005
2	.0046	0043	0040	0037	0034	0031	0029	0027	0025	0023
3	.0140	0131	0123	0115	0107	0100	0093	0087	0081	0076
4	.0319	0302	0285	0269	0254	0240	0226	0213	0201	0189
5	.0581	0555	0530	0506	0483	0460	0439	0418	0398	0378

λ=	9.1	9.2	9.3	9.4	9.5	9.6	9.7	9.8	9.9	10.0
$x=$ 6	.0881	0851	0822	0793	0764	0736	0709	0682	0656	0631
7	.1145	1118	1092	1064	1037	1010	0983	0955	0928	0901
8	.1302	1286	1269	1251	1232	1212	1191	1170	1148	1126
9	.1317	1315	1311	1306	1300	1293	1284	1273	1263	1251
10	.1198	1210	1219	1228	1235	1241	1245	1248	1250	1251
11	.0991	1012	1031	1049	1067	1083	1098	1112	1125	1137
12	.0752	0776	0799	0822	0844	0866	0888	0908	0929	0948
13	.0526	0549	0572	0594	0617	0640	0662	0685	0707	0729
14	.0342	0361	0380	0399	0419	0439	0459	0479	0500	0521
15	.0208	0221	0235	0250	0265	0281	0297	0313	0330	0347
16	.0118	0127	0137	0147	0158	0169	0180	0192	0204	0217
17	.0063	0069	0075	0081	0088	0095	0103	0111	0119	0128
18	.0032	0035	0039	0042	0046	0051	0055	0060	0065	0071
19	.0015	0017	0019	0021	0023	0026	0028	0031	0034	0037
20	.0007	0008	0009	0010	0011	0012	0014	0015	0017	0019
21	.0003	0003	0004	0004	0005	0006	0006	0007	0008	0009
22	.0001	0001	0002	0002	0002	0002	0003	0003	0004	0004
23		0001	0001	0001	0001	0001	0001	0001	0002	0002
24								0001	0001	0001

(Printed March 9 1906.)

Fig. 1 Normalcurve.

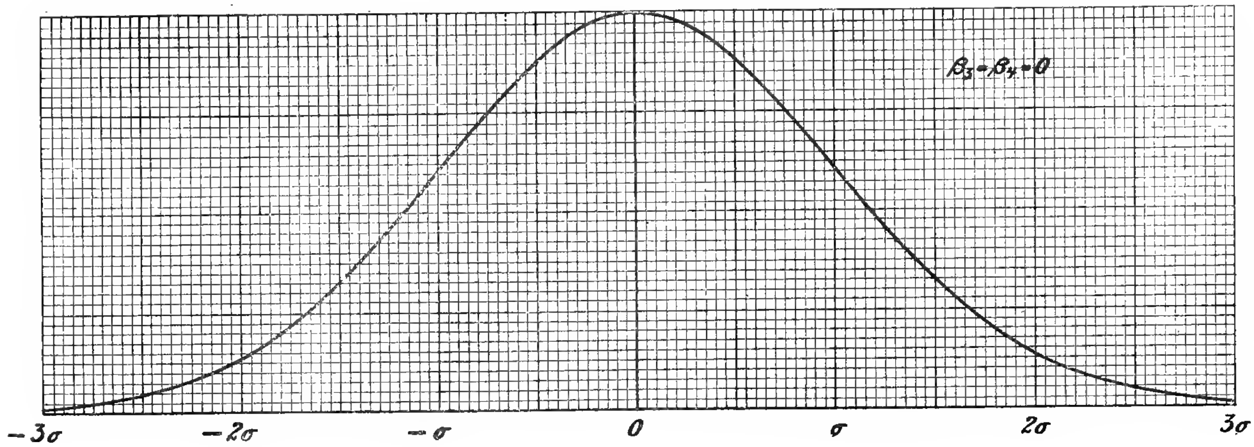


Fig. 2 Frequency curve with positive skewness.

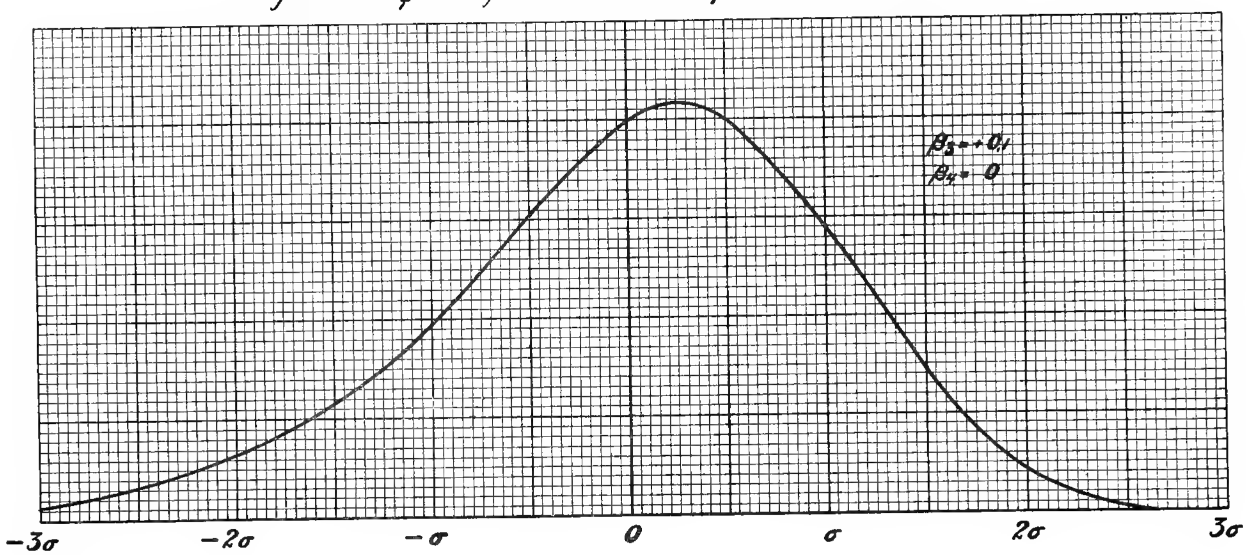


Fig. 3 Frequency curve with positive skewness.

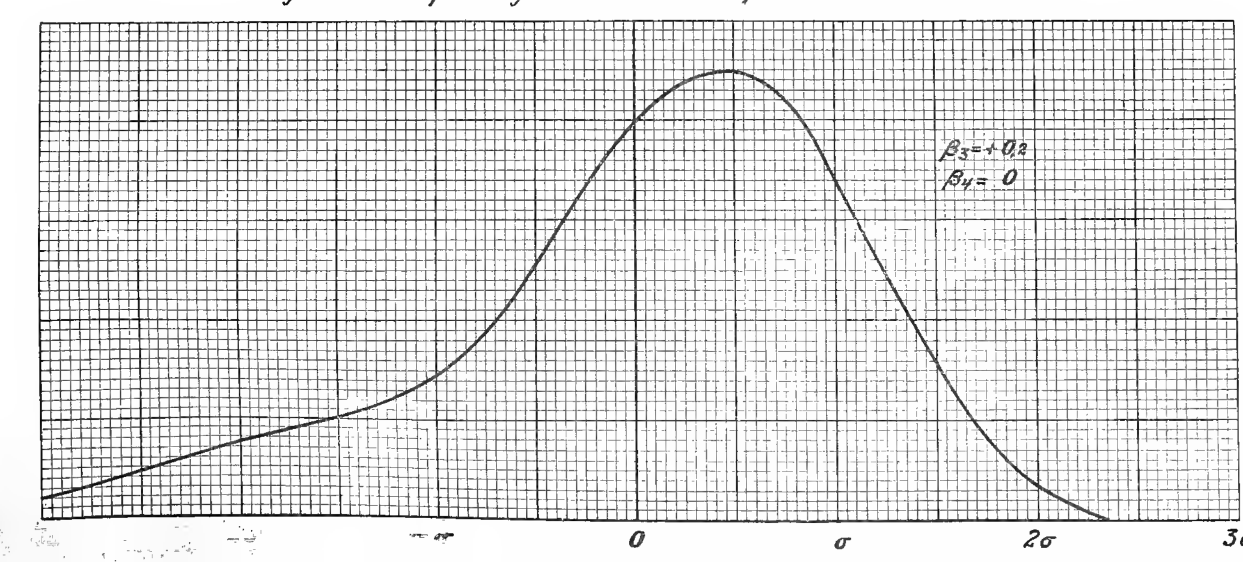


Fig. 4 Frequency curve with positive excess.

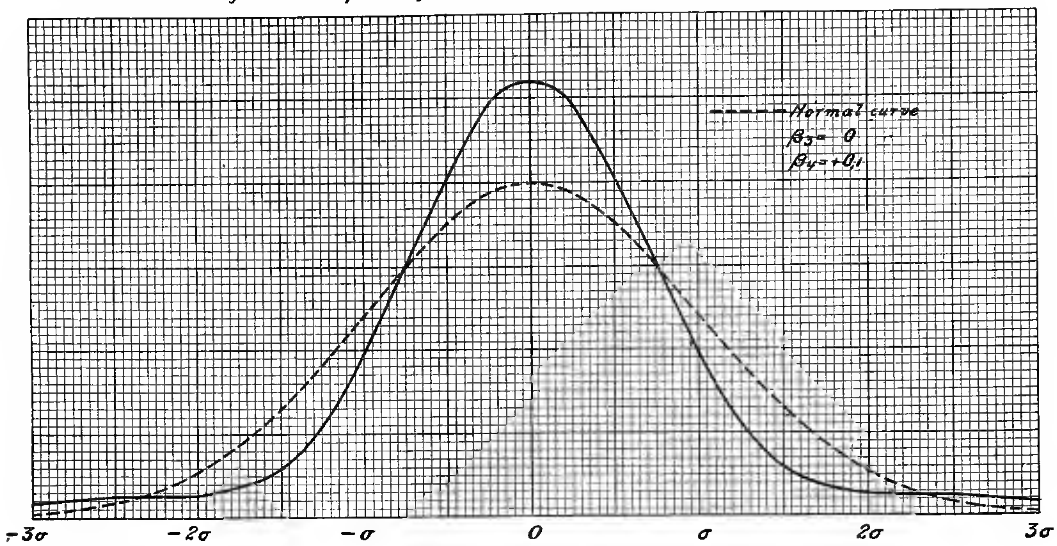


Fig. 5 Frequency curve with negative excess.

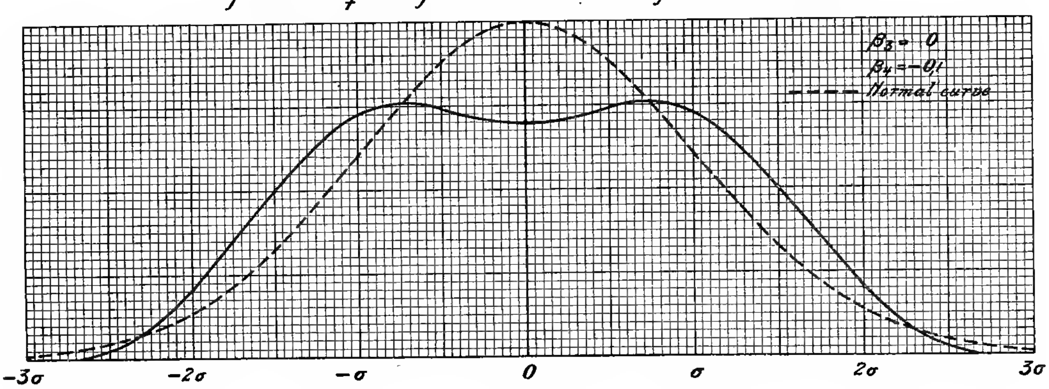


Fig. 6 Frequency of glands in the leg of female swine. (Davenport).

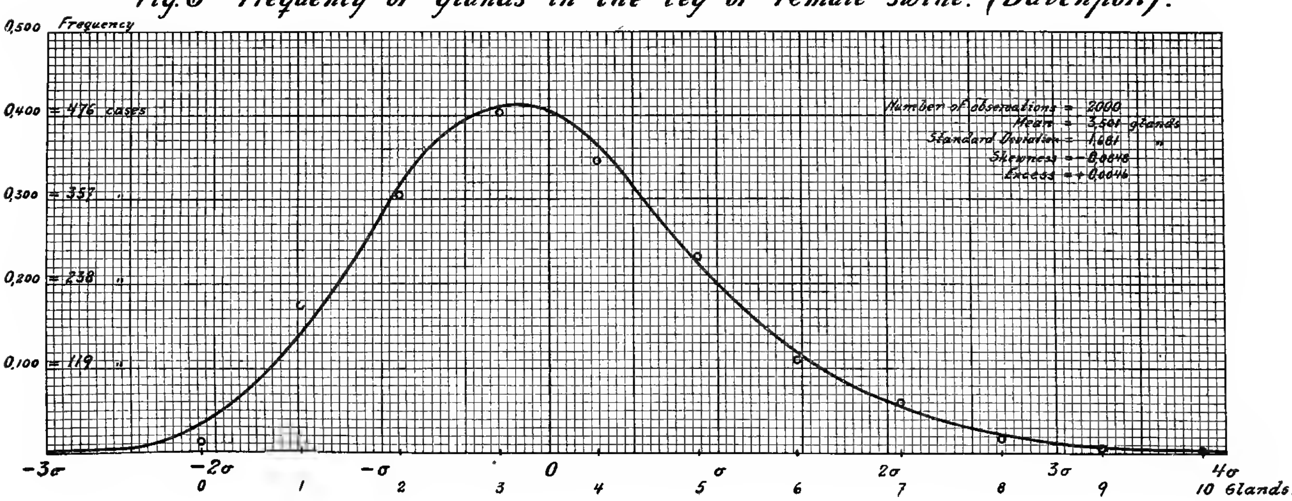


Fig. 7 Frequency of stigmatic bands of Papaver (Charlier).

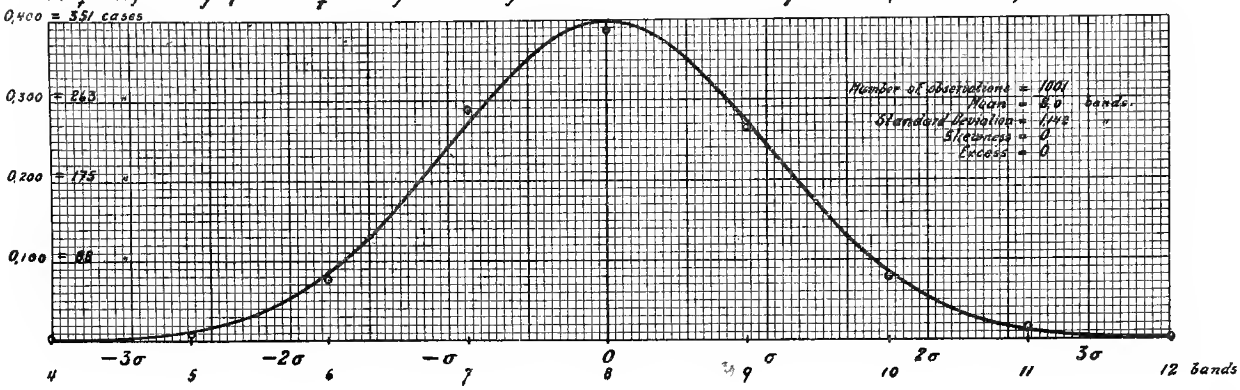


Fig. 8 Line A of brown beans (Johannsen)

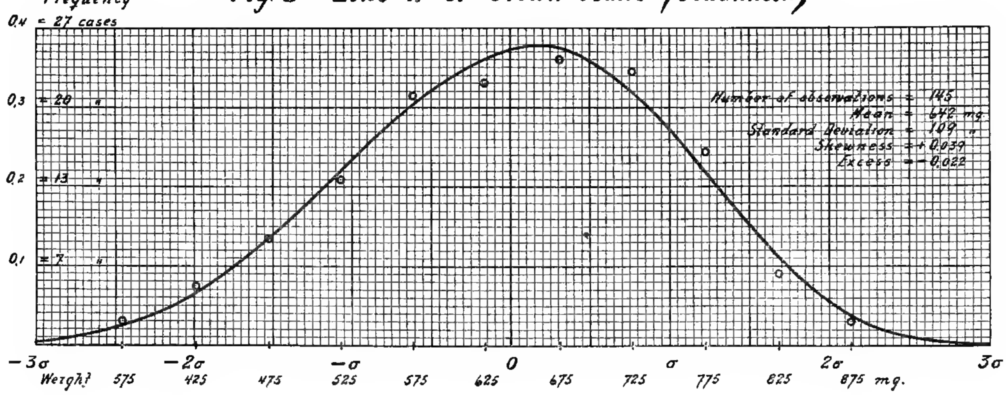


Fig. 9 Line G of brown beans (Johannsen)

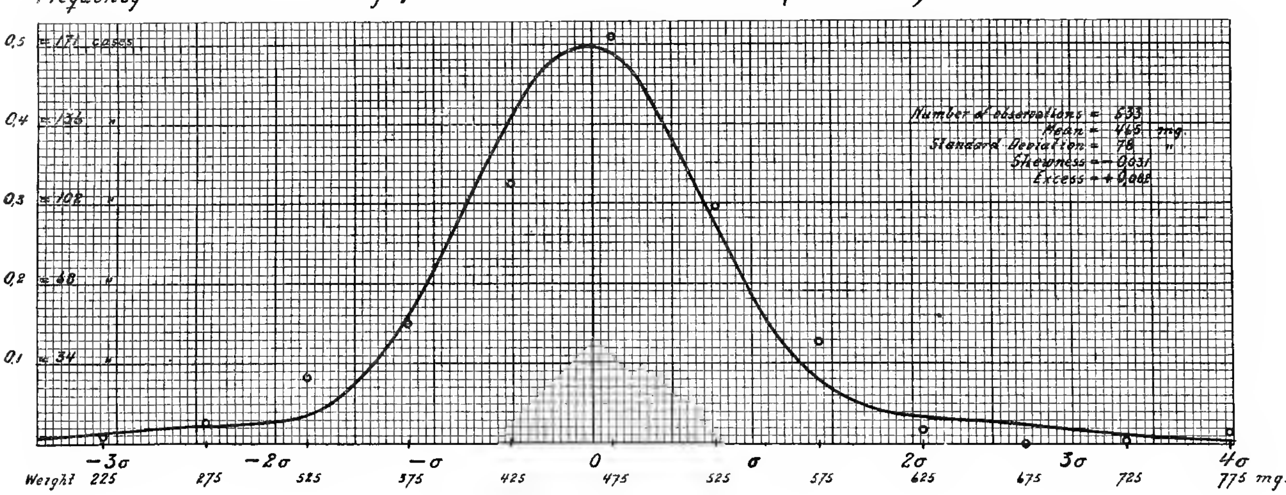


Fig. 10 Line 0 of brown beans (Johannsen).

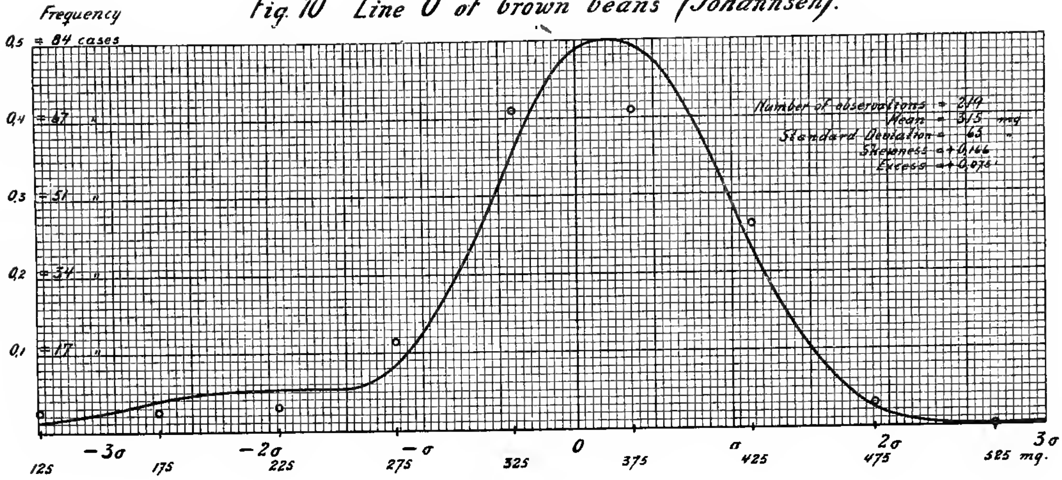


Fig. 11 Cephalic Index of Swedish recruits (Retzius and Fürst).

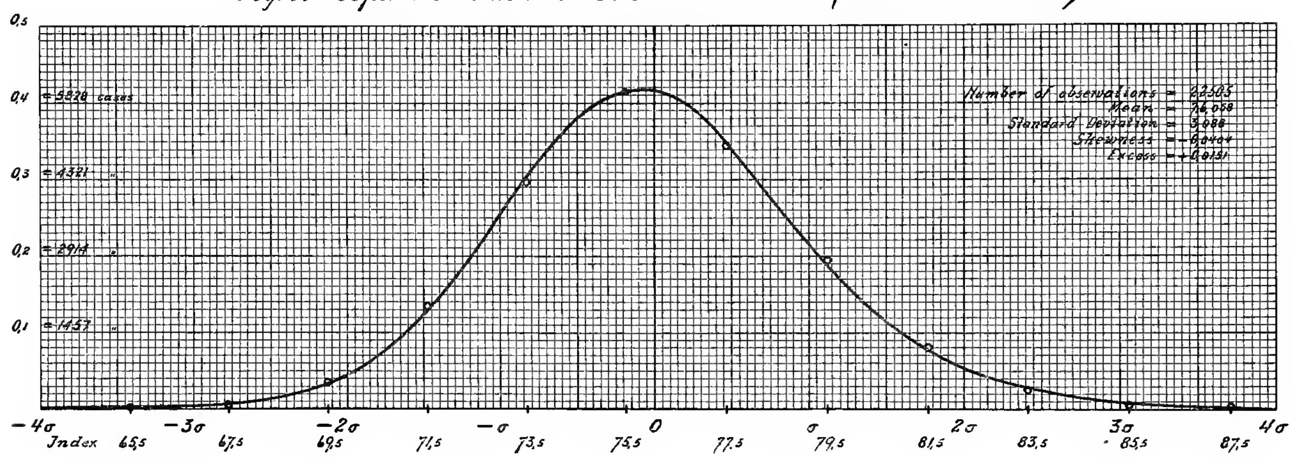


Fig. 12 Typhoid Fever in Lund 1905 (Ryberg).

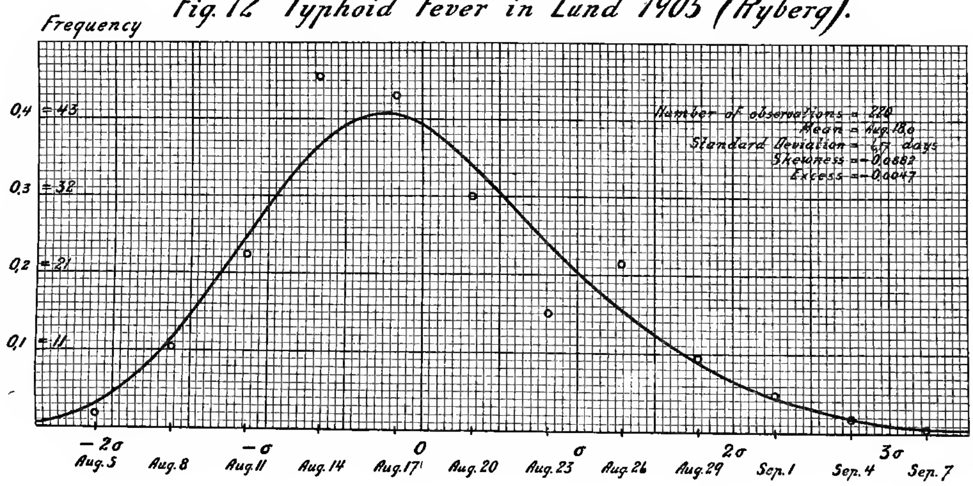


Fig. 13 Frequency curves of type B

$y = \psi_\lambda(x)$.

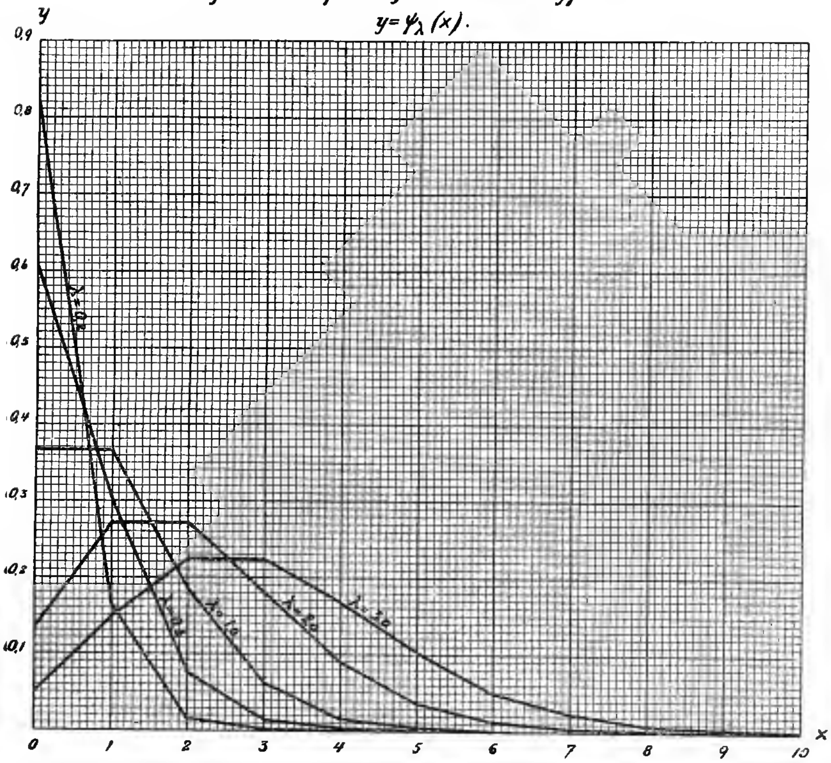


Fig. 14. Petals of Ranunculus bulbosus.

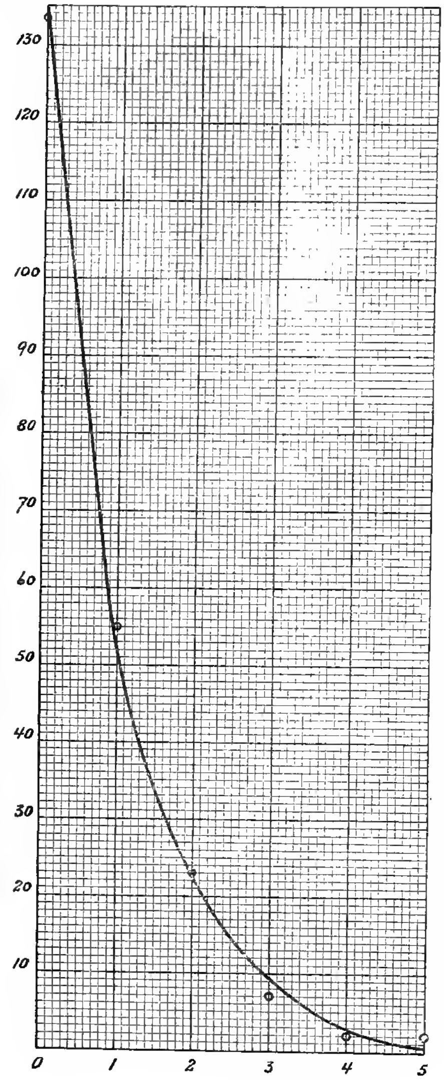


Fig. 16 Glands of swine treated as a B-curve.

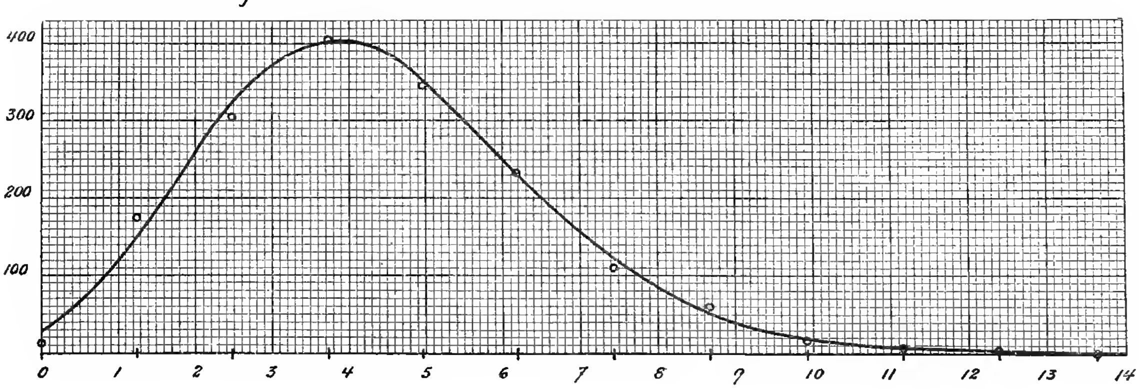


Fig. 15 Failing percentage of barley (Johannsen)

$F(x) = \psi(0{,}720x - 0{,}395)$.

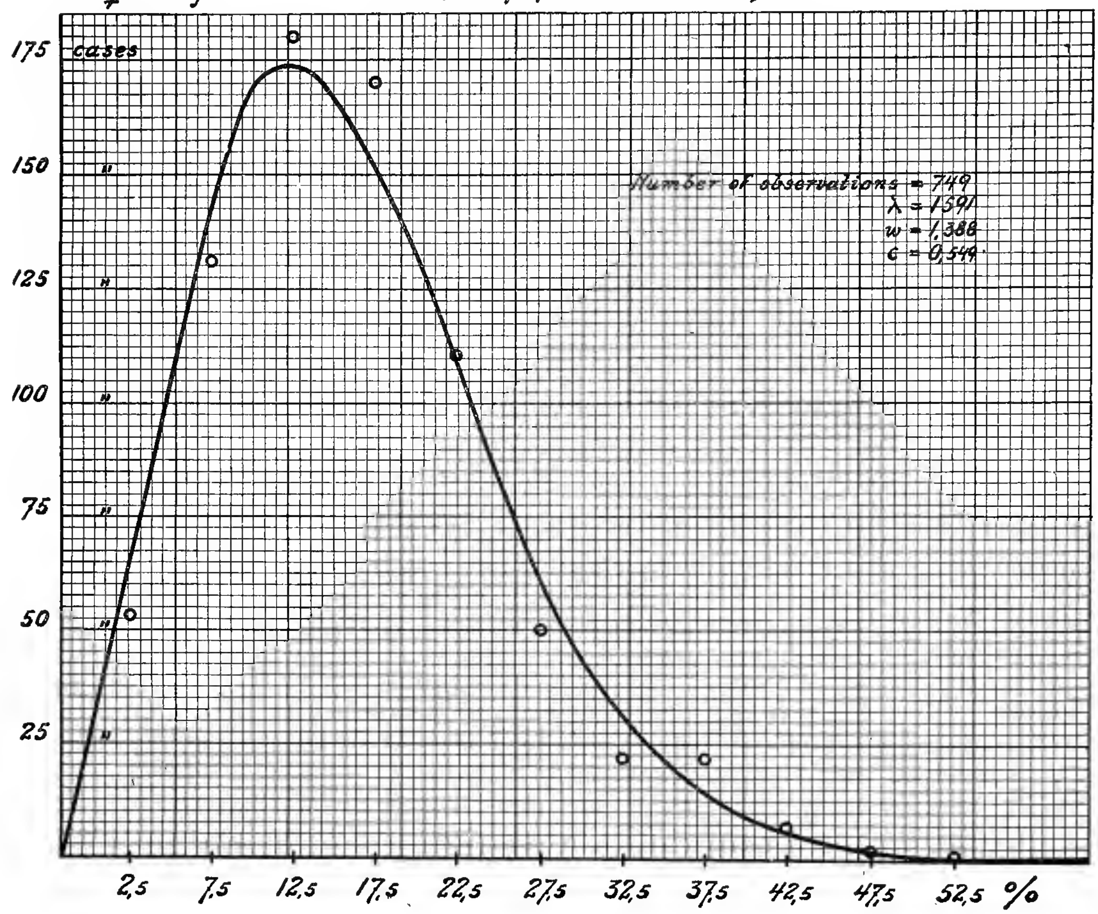
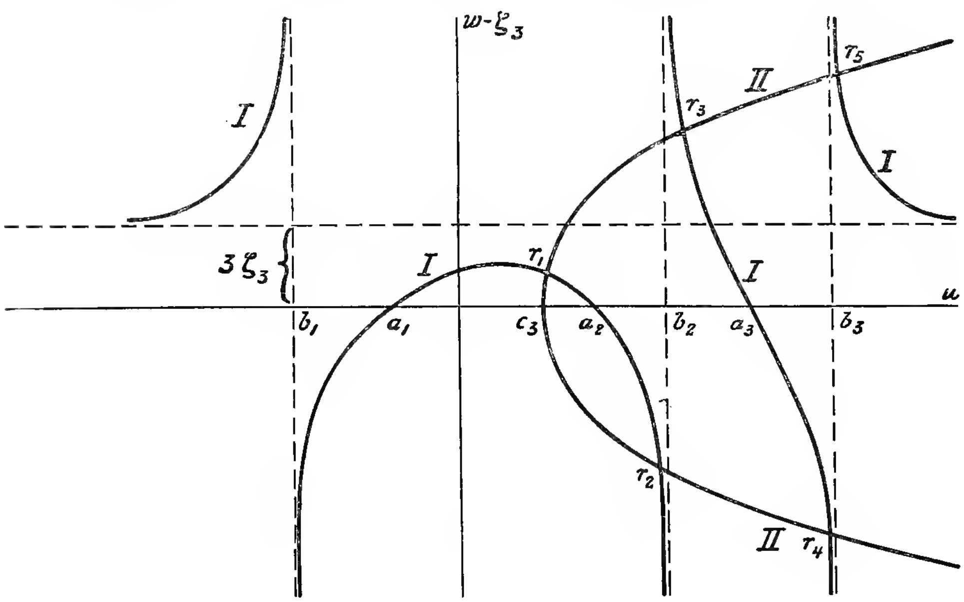


Fig. 17. Dissection of Frequency Curves.



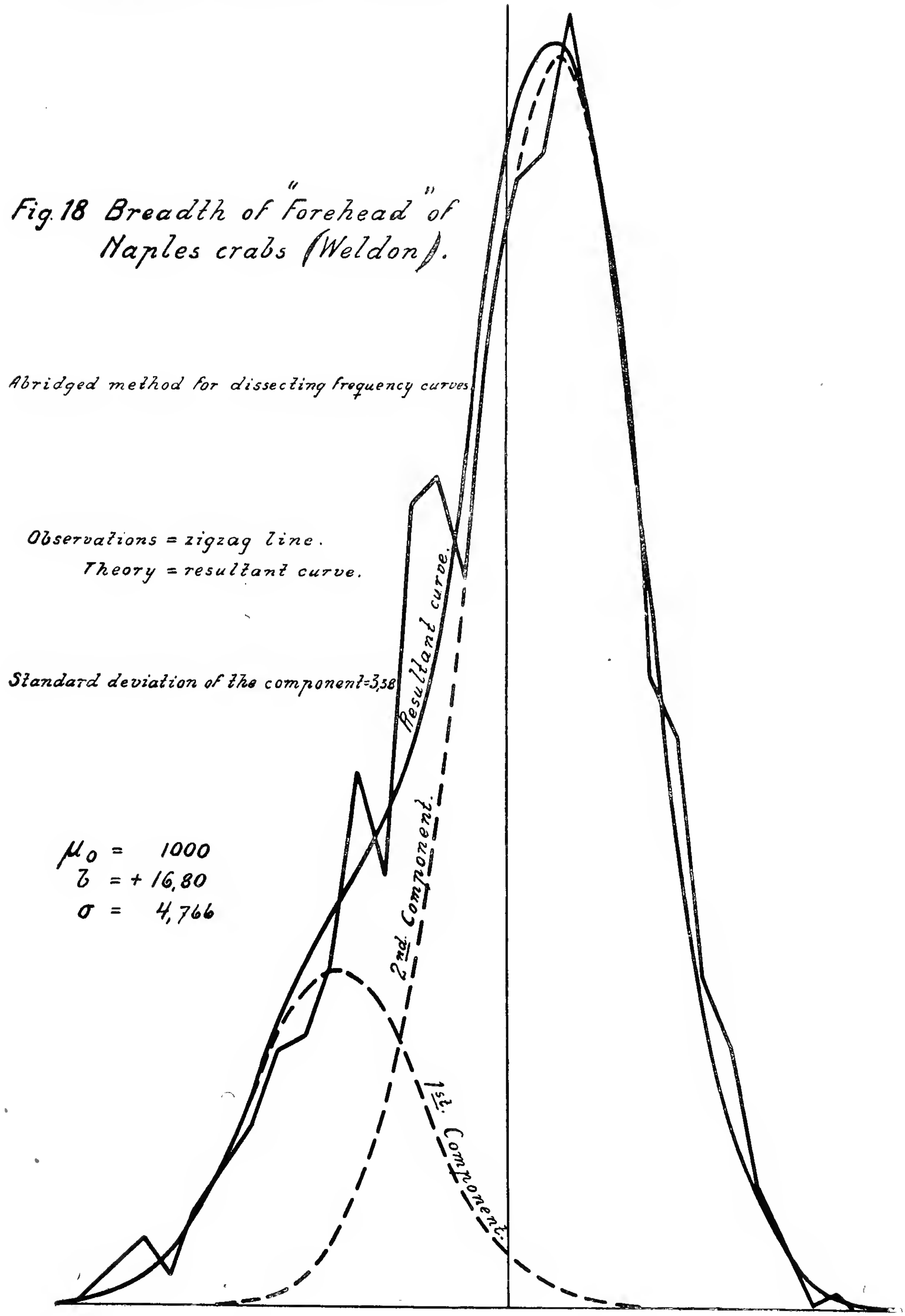


Fig. 18 Breadth of "Forehead" of Naples crabs (Weldon).

Fig. 19.

DISTRIBUTION OF FREQUENCY OF RAY FLOWERS OF 1015 SAMPLES OF CHRYSANTEMUM SEGETUM.